Brownian Bridge Method

The Brownian bridge algorithm generates points on a path sequentially, starting from the initial point and progressing toward the end. However, the symmetry of the formula for the intermediate point with respect to the initial and final points and the recursive nature of the algorithm suggest that the generator may work better.

If every new point is placed in approximately the middle of the corresponding subinterval. If a variance reduction is essential, this modification may cause a significant improvement. Although the variance reduction is not an issue in the context of the stress testing, it is worth considering such a modification of the algorithm in the future releases.

The Brownian bridge algorithm has been implemented for stress testing within the Risk Management framework. It is used for generation of multidimensional random paths whose initial and ending points are predetermined and fixed.

The Brownian Bridge algorithm belongs to the family of Monte Carlo or Quasi-Monte Carlo methods with reduced variance. It generates sample paths which all start at the same initial point and end, at the same moment of time, at the same final point.

Consider a stochastic process of the form

$$dx = \sigma dz \qquad (1)$$

where z is a standard Brownian motion.

A random jump from a point x_0 at time 0 to a point x_T at time T can thus be expressed as

$$x_T = x_0 + \sigma\sqrt{T}z_T \quad (2)$$

Given the initial point x_0 and the final point x_T, the Brownian bridge algorithm provides the formula for an intermediate point x_t:

$$x_t = (1-\gamma_t)x_0 + \gamma_t\, x_T + \sigma\sqrt{\gamma_t(1-\gamma_t)T}z_t \quad (3)$$

where

$$\gamma_t = \frac{t}{T}$$

Once x_t has been determined, formula (3) can be used for the subintervals (0, t) and (t,T). The recursive implementation of the algorithm produces a random path connecting x_0 and x_T.

If a random process x_t is described by eq. (1), the conditional probability that x_t takes on the value x_f at some final time t_f provided that at the starting time t_i it was x_i and at an intermediate time t it was x is

$$P(x_f, t_f | x, t | x_i, t_i) = \frac{1}{2\pi\sigma^2\sqrt{(t_f - t)(t - t_i)}}\exp[-\frac{(x_f - x)^2}{2\sigma^2(t_f - t)} - \frac{(x - x_i)^2}{2\sigma^2(t - t_i)}] =$$

$$\frac{1}{\sqrt{2\pi(t_f - t_i)}\,\sigma}\exp[-\frac{(x_f - x_i)^2}{2\sigma^2(t_f - t_i)}]\times$$
$$\frac{1}{\sqrt{2\pi}\,\sigma}\sqrt{\frac{(t_f - t_i)}{(t_f - t)(t - t_i)}}\exp\{-\frac{(t_f - t_i)}{2\sigma^2(t_f - t)(t - t_i)}[x - (x_f\frac{t - t_i}{t_f - t_i} + x_i\frac{t_f - t}{t_f - t_i})]^2\} \quad (4)$$

Eq. (4) is a product of the two factors: the first is the probability of getting to point x_f at time t_f starting from point x_i at time t_i, and the second is the probability of passing through point x at time t given those initial and final points.

It is the second factor in formula (4) that gives the distribution of points on the paths connecting fixed initial and final points, i.e. those generated by the Brownian bridge algorithm. In particular, according to eq. (4) for any given time t between t_f and t_i the distribution of points x should be normal with the mean

$$\bar{x}(t) = x_f \frac{t - t_i}{t_f - t_i} + x_i \frac{t - t_f}{t_f - t_i} \tag{5}$$

and variance

$$\mathrm{var}_x(t) = \sigma^2 \frac{(t_f - t)(t - t_i)}{(t_f - t_i)} \qquad (6).$$

Note that the function $\bar{x}(t)$ represents a straight line connecting the initial and final points. Note also that eq. (6) can be cast into a slightly different form:

$$\sigma^2 = \frac{(t_f - t_i)}{(t_f - t)(t - t_i)} \mathrm{var}_x(t) \tag{6a}$$

The Brownian Bridge algorithm is also very useful for valuing exotic derivatives via Monte Carlo approach. Most exotic products have callable and barrier features. Both callable and barrier characterizations can terminate a trade earlier. You can find barrier feature at https://finpricing.com/lib/EqBarrier.html