

# Martingale Preserving Tree Method

An important feature of the popular three factor trinomial tree is that it uses a deterministic approximation of the interest rates for constructing the stock tree. The preservation of the martingale property of the stock price is thus not guaranteed. and may potentially represent a problem.

As a more general observation, one should mind the limitations of the short term rates models such as the Hull-White model or the Black-Karasinski model. It is well known that in the Hull-White model the interest rate may become negative, which breaks the no-arbitrage condition.

On the Hull-White tree, the negative rates occur for sufficiently long terms, or even for medium terms when a fine time discretization is used. The problem is mitigated by the fact that the rates become negative in the lower part of the tree, whose contribution is small due to the low probability of reaching that part of tree. However, the experience shows that the Hull-White model is not very good for terms exceeding 10 years.

Another kind of problem arises when one uses a log-normal models of the short rates, such as the Black-Karasinski model, in conjunction with log-normal stock models. The stock price grows exponentially with the rate which, in turn, is an exponential function of the random factor.

As a result, there is a non-zero probability that the stock price turns infinite within a finite time. One may argue that when those large stock prices are discounted back with the appropriate rates they should give reasonable contribution to the present value.

In practice, however, one should take into account the truncation errors of the numerical procedures, which may cause large distortions in such a discounting. Just as in the case of the Hull-White model, the perturbing effect may be diminished by a low probability of reaching the upper part of the tree. However, we are not aware of systematic studies of such truncation errors in the Black-Karasinski tree and recommend caution in the tree use for long term securities. Alternative approaches to modeling interest rates are worth to be considered.

We propose a two-factor tree model that implements the Hull-White and Black-Karasinski models. The new tree model does preserve the martingale property of the stock for sufficiently long terms (with accuracy better than  $10^{-8}$  for terms of at least 10 years).

The three factor model assumes that under the risk neutral probability measure the stock price follows the stochastic process

$$dS_t = S_t \left( [r_t - q_S] dt + \sigma_S dW_t^S \right), \quad (1)$$

and the interest rate process is governed by the Ho-Lee model:

$$dr_t = \theta(t)dt + \sigma_R dW_t^r, \quad (2)$$

where  $q_S$  is the dividend rate. The Brownian motions  $W_t^r$  and  $W_t^S$  are supposed to be correlated. Next, the model makes an approximation and assumes that  $r_t$  in eq. (1) is a deterministic variable  $r_S$  such that

$$e^{\int_0^t r_s ds} = \frac{1}{P(0,t)}, \quad (3)$$

where  $P(0,t)$  is the price at time 0 of zero coupon bond maturing at  $t$ . As a result, the random walk parts of the  $S$  and  $r$  are only coupled through the linear stochastic terms in equations (1) and (2). This makes it possible to construct two individual one-factor tree, one for the  $S$  process and the other for the  $r$  process, which are then combined in a two-factor tree.

The coupon is the nominal annual rate of interest that is paid to the holder on a regular basis. It is usually expressed as a percentage of the face value (coupon rate: <https://finpricing.com/lib/FiBondCoupon.html>). The coupon rate is either fixed or variable. The coupon rate is given as an annualized percentage of the face value.

The probabilities of the joint Brownian motion on the combined tree are adjusted according to the correlation between  $W^r_t$  and  $W^S_t$ . This approach produces an efficient implementation, since it uses only two one-dimensional trees. It has a significant drawback, however: the martingale property of the stock price is not preserved.

The new model assumes the rate process to be governed by the SDE

$$dx_t = (\theta(t) - ax_t)dt + \sigma_R dW_t^r, \quad (4)$$

where  $x$  is  $r$  for the Hull-White model, or  $\log(r)$  for the Black-Karasinski model.

The stock price is supposed to follow the stochastic process

$$dS_t = S_t \left( [r_t - q_s] dt + \sigma_s \sqrt{1 - \rho^2} dW_t^S + \sigma_s \rho dW_t^r \right), \quad (5)$$

where  $W_t^r$  and  $W_t^S$  are independent Brownian motions and  $\rho$  is the correlation coefficient.

Since eq. (4) is independent of eq. (5), the interest rate process is a one-factor process and the corresponding tree is constructed in the standard manner.

Eq. (5) depends on two factors, and the equity tree is built as a genuine two-dimensional tree. Its r-projection replicates the above interest rate tree, with the same values of  $r$  and transition probabilities at the corresponding nodes in the r-dimension. The distribution of the stock values is calculated from eq. (5) and the overall transition probabilities are adjusted so as to preserve the martingale property and, if possible, to match the variance. The new trees allow variable time steps.