

# Pricing CMS Spread Option

A constant maturity swap (CMS) spread option makes payments based on a bounded spread between two index rates (e.g., a GBP CMS rate and a EURO CMS rate). We assume that both the forward GBP and EURO CMS rates follow geometric Brownian motion under their respective T-forward measures.

To accurately value the option we require the joint density, under the EURO (domestic) T-forward measure, of the respective CMS rates and the bond's price. For an approximate value, however, we adjust the forward GBP CMS rate against a GBP denominated zero-coupon bond. The resulting CMS rate is then quanto adjusted. The forward EURO CMS rate is also adjusted for a delayed payment.

The GBP CMS rate is calculated from a 15 year swap with semi-annual, upfront payments, while the EURO CMS rate is based on a 15 year swap with annual, upfront payments.

Let

- $T$  be a CMS rate fixing time,
- $T + \Delta$  be a payment time,
- $G$  and  $E$  respectively denote the GBP and EURO CMS rates at the fixing time,
- $S = 2G - E$  be a CMS rate spread.

Then, at time  $T + \Delta$ , the buyer must pay a spread,  $S$ , bounded below and above by 0% and 10% respectively, multiplied by the accrual period,  $\Delta$ ; formally the payoff is given by

$$\Delta \min ( 10\%, \max (0\%, S) ).$$

We assume that both the forward GBP and EURO CMS rates follow geometric Brownian motion under their respective  $T$ -forward measures. Here respective initial forward CMS rates are calculated. The forward rates are then convexity adjusted from respective parallel bonds specified using

- the same payment times as the underlying CMS,
- coupon equal to the forward CMS rate,
- initial bond yield equal to the forward CMS rate.

Let

- $G_t$  be a GBP swap rate at time  $t$ ,
- $U_t$  be a EURO swap rate at time  $t$ ,
- $S_T = 2G_T - U_T$  be a spread between the two swap rates above.

Then

$$\begin{aligned} \min(\max(S_T, 0), 10\%) &= -\max(-\max(S_T, 0), -10\%), \\ &= -(\max(0, \max(S_T, 0) - 10\%) - \max(S_T, 0)), \\ &= \max(S_T, 0) - \max(0, \max(S_T, 0) - 10\%), \\ &= \max(S_T, 0) - (-10\% + \max(10\%, \max(S_T, 0))), \\ &= \max(S_T, 0) + 10\% - \max(10\%, \max(S_T, 0)), \\ &= \max(S_T, 0) + 10\% - \max(10\%, S_T), \\ &= \max(S_T, 0) + 10\% - (10\% + \max(0, S_T - 10\%)), \\ &= \max(S_T, 0) - \max(0, S_T - 10\%). \end{aligned}$$

Furthermore, since  $S_T = \max(S_T, 0) - \max(-S_T, 0)$ ,

$$\min(\max(S_T, 0), 10\%) = S_T + \max(-S_T, 0) - \max(0, S_T - 10\%).$$

The buyer must pay  $\Delta S_T$  at time  $T + \Delta$  where  $\Delta$  (with  $\Delta > 0$ ) is an accrual period. Let

- $E^{T+\Delta}$  denote the domestic  $T + \Delta$ -forward probability measure,
- $E^T$  denote the domestic  $T$ -forward probability measure,
- $E^Q$  denote the domestic risk-neutral probability measure.

Then

$$\begin{aligned} P(0, T + \Delta)E^{T+\Delta}(S_T) &= E^Q\left(\frac{S_T}{\beta_{T+\Delta}}\right), \\ &= E^Q\left(E^Q\left(\frac{S_T}{\beta_{T+\Delta}} \middle| F_T\right)\right), \\ &= E^Q\left(S_T E^Q\left(\frac{1}{\beta_{T+\Delta}} \middle| F_T\right)\right), \\ &= E^Q\left(\frac{S_T P(T, T + \Delta)}{\beta_T}\right), \\ &= P(0, T)E^T(S_T P(T, T + \Delta)), \end{aligned}$$

where  $P(t, \tau)$  is the price at time  $t$  of a zero-coupon bond that matures at time  $\tau$  with face value of 1 Euro.

We assume that the forward CMS rate processes,  $\{G_t \mid 0 \leq t \leq T\}$  and  $\{U_t \mid 0 \leq t \leq T\}$ , satisfy, under domestic T-forward measure, the respective SDEs

$$dG_t = G_t \sigma_G dW_t^G,$$

$$dU_t = U_t \sigma_U dW_t^U,$$

where

- $\sigma_G$  and  $\sigma_U$  are respective constant volatilities,
- $\{W_t^G \mid t \geq 0\}$  and  $\{W_t^U \mid t \geq 0\}$  are standard Brownian motions with instantaneous correlation,  $\rho_{GU}$ .

Here  $G_o$  is determined from respective convexity and quanto adjustments, while  $U_o$  is determined from a convexity adjustment.

Let  $y_t$  satisfy

$$\frac{1}{1 + \Delta y_t} = \frac{P(t, T + \Delta)}{P(t, T)}.$$

Observe that, under domestic  $T$ -forward probability measure,

$$\begin{aligned}
E^T \left( \frac{1}{1 + \Delta y_t} \middle| F_\tau \right) &= E^T \left( \frac{P(t, T + \Delta)}{P(t, T)} \middle| F_\tau \right), \\
&= \frac{P(\tau, T + \Delta)}{P(\tau, T)}, \\
&= \frac{1}{1 + \Delta y_\tau},
\end{aligned}$$

so that  $\left\{ \frac{1}{1 + \Delta y_t} \middle| 0 \leq t \leq T \right\}$  is a martingale under this measure. We assume that the process

$\{y_t | 0 \leq t \leq T\}$  satisfies, under domestic T-forward measure, the SDE

$$dy_t = y_t \sigma_y dW_t^y$$

where

- $\sigma_y$  is a constant volatility,
- $\{W_t^y | 0 \leq t \leq T\}$  is a standard Brownian motion with respective pairwise correlation to  $\{W_t^U | 0 \leq t \leq T\}$  and  $\{W_t^G | 0 \leq t \leq T\}$ ,  $\rho_{YU}$  and  $\rho_{YG}$ ,
- $\frac{1}{1 + \Delta y_0} = \frac{P(0, T + \Delta)}{P(0, T)}$ .

Assuming that  $\{y_t | 0 \leq t \leq T\}$  satisfies the SDE above, however, the martingale property above does not hold.

In order to satisfy the martingale condition

$$E^T \left( \frac{1}{1 + \Delta y_T} \right) = \frac{P(0, T + \Delta)}{P(0, T)},$$

we convexity adjust the initial forward bond yield,  $y_0$ .

Observe that

$$\begin{aligned} X_0 P^f(0, T) E^{T_f} \left( \frac{G_T}{1 + \Delta y_T^f} \right) &= P^d(0, T) E^{T_d} \left( \frac{G_T X_T}{1 + \Delta y_T^f} \right), \\ &= P^d(0, T) E^{T_d} \left( \frac{G_T x_T}{1 + \Delta y_T^d} \right), \end{aligned}$$

where

- $X_t$  is the spot exchange rate from foreign (GBP) currency into domestic (EURO) currency,
- $x_t$  is the forward exchange rate, at seen at time  $t$ , from foreign currency into domestic, at the forward time  $T + \Delta$ .

From the above the correlation, under foreign T-forward measure, between the Brownian motions respectively driving the foreign bond yield and GBP swap rate processes constrains the correlation between the Brownian motions driving the Euro bond yield and forward GBP CMS rate processes under domestic T-forward measure. Bond yield is computed from bond price (see <https://finpricing.com/lib/FiZeroBond.html>).

To evaluate the expression above, we require the SDE followed by the forward exchange rate process under domestic  $T$ -forward measure. This is derived by changing measure from foreign  $T + \Delta$ -forward measure, to domestic risk-neutral measure, to domestic  $T$ -forward measure.