Pricing Single Currency Bermudan Swaption

The underlying security of a single currency Bermudan swaption is an interest-rate swap, which is specified by respective payer and receiver legs. Each of the legs above can pay a fixed rate, Libor or CMS rate. The owner of the Bermudan swaption can choose to enter into the swap above at certain pre-defined exercise times; upon exercise, the owner

- must pay all payer-leg quantities that reset on or after the exercise time, and
- will receive all receiver-leg quantities that reset on or after the exercise time.

The pricing method is based on Jamshidian's Libor rate model (i.e., where Libor rates are modeled simultaneously under the spot Libor measure). Furthermore, we value a Bermudan swaption based on the Monte Carlo technique presented by Longstaff and Schwartz towards American style pricing.

Let $T_1,\ldots,T_N$, where $0 < T_1 < ... < T_N$, be common Libor reset points. Here we assume that all interest rate reset and Bermudan exercise times belong to the set of common reset points, $\{T\}$.

We consider an interest-rate swap consisting of respective receiver and payer legs. Here the payer leg is specified by

- reset points, $t_1,..., t_M$, where $\{t\} \subseteq \{T\}$ and $t_1 < ... < t_M$,
- an amount, $R(t_{i+1} - t_i)$, payable at time t_{i+1}, for $i = 1,\ldots,M-1$.

Here R can be a fixed rate, a Libor or a CMS rate. In the case of an n-period Libor rate,

$$R = \frac{1}{\sum_k \Delta_{p+k-1}} \left(\frac{1}{P(T_p, T_{p+n})} - 1 \right)$$

where

- $t_i = T_p$,
- $\Delta_i = T_{i+1} - T_i$,
- $P(t,T)$ is the price at time t of a zero-coupon bond, which matures at T and has $1 face value.

In the case of a single period Libor rate, for example, $R = \frac{1}{\Delta_p} \left(\frac{1}{P(T_p, T_{p+1})} - 1 \right)$.

For an m-period CMS rate with frequency, f (where f is a whole number of consecutive common reset periods),

$$R = \frac{1 - P(T_p, T_{p+fm})}{\sum_k \left(\sum_l \Delta_{p+f(k-1)+l-1} \right) P(T_p, T_{p+fk})}$$

For example, if $f = 1$ (i.e., the CMS has reset times that correspond to consecutive common reset points), then $R = \frac{1 - P(T_p, T_{p+m})}{\sum_k \Delta_{p+k-1} P(T_p, T_{p+k})}$.

Let the receiver leg be similarly defined with respect to the reset points, $\tau_1, ..., \tau_l$.

We now consider exercise points, $t'_1,\ldots,t'_p$, such that $\{t'\}\subseteq\{T\}$. Here the Bermudan swaption can be exercised at any point belonging to $\{t'\}$. If the option is exercised at time t'_k, for some $k\in\{1,\ldots,p\}$, then

- the owner must pay all payer-leg quantities that set at points t_i such that $t_i \geq t'_k$, and
- the owner receives all receiver-leg quantities that set at points τ_i such that $\tau_i \geq t'_k$.

We model Libor rates under the spot Libor measure, which has numeraire process,

$$N_{T_i} = \prod_{j=1}^{i} \frac{1}{P(T_{j-1},T_j)},$$

where $P(t,T)$ denotes the price at time t of a zero coupon bond with maturity of T (ref https://finpricing.com/lib/FiBondCoupon.html).

Let $[t]$ denote the integer, i, such that $T_{i-1} < t \leq T_i$. Under the spot Libor measure, we assume that, for $i=1,\ldots,N-1$ and $0 \leq t \leq T_i$,

$$dL_i = L_i\,\vec{\sigma}(t,T_i)\bullet\left(\sum_{j=[t]}^{i}\frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\vec{\sigma}(t,T_j)dt + d\vec{W}\right), \qquad (3.1.1)$$

where

- $\vec{W}\in\Re^4$ is a vector of uncorrelated, standard Brownian motions,
- $\vec{\sigma}\in\Re^4$ is a time deterministic volatility vector, which we define below.

We consider a volatility vector of the form

$$\vec{\sigma}(t,T) = \vec{\psi}(T-t)\upsilon(t,T).$$

Here

$$\begin{aligned}
\psi_1(t) &= c, \\
\psi_2(t) &= \sqrt{\frac{1-c^2}{1+(t/a)^{\alpha}}}, \\
\psi_3(t) &= \sqrt{(1-c^2)\left(1-\frac{1}{1+(t/a)^{\alpha}}\right)\left(\frac{1}{1+(t/b)^{\beta}}\right)}, \\
\psi_4(t) &= \sqrt{1-[\psi_1(t)]^2-[\psi_2(t)]^2-[\psi_3(t)]^2}.
\end{aligned}$$

Furthermore

$$\upsilon(t,T) = e^{\sum_i \sum_j a_{ij} \Phi_i(x_1) \Phi_j(x_2)}$$

where

$$\begin{aligned}
x_1 &= -1 + 2e^{-(T-t)/\tau_1}, \\
x_2 &= -1 + 2e^{-T/\tau_2}.
\end{aligned}$$

Here

$$\Phi_i(x) = \cos\left(i\cos^{-1} x\right)$$

denotes a Chebyshev polynomial of the first kind.

In the above, the parameters

$$a, b, c, \alpha, \beta, \tau_1, \tau_2, \left(a_{ij}\right), \tag{3.1.2}$$

are determined from calibration.

Let t' be an exercise time. Suppose that the payer leg has n future reset times, $t_1,..., t_n$, such that $t_1,...,t_n \geq t'$. Furthermore assume that an interest rate quantity, p_i, for $i = 1,..., n-1$, sets at t_i and is paid at t_{i+1}. Similarly assume that the receiver leg has m future reset times, $\tau_1,..., \tau_m$, where $\tau_1,...,\tau_m \geq t'$. Furthermore assume that an interest rate quantity, r_i, for $i = 1,..., m-1$, sets at τ_i and is received at τ_{i+1}.

The payoff from a European option to enter into the swap at t' is then given by

$$\max\left(N_{t'} \sum_{i=1}^{m-1} E\left(\frac{r_i}{N_{\tau_{i+1}}} \middle| F_{t'} \right) - N_{t'} \sum_{i=1}^{n-1} E\left(\frac{p_i}{N_{t_{i+1}}} \middle| F_{t'} \right),\ 0 \right) \tag{3.2.1}$$

where

- F_t denotes the sigma algebra induced by Brownian motion up to time t,
- E denotes expectation with respect to the spot Libor measure.

We price a Bermudan style swaption using a Monte Carlo technique, which is based on the approach proposed by Longstaff and Schwartz towards American style pricing using simulation. In particular, at every exercise time, we must solve a linear least squares problem, and then decide whether to exercise the option.

We assume that the short interest-rate, r, follows a risk-neutral Hull-White process, of the form

$$dr_t = (\theta_t - ar_t)dt + \sigma dW_t \, ,$$

where

- a is a constant mean reversion parameter,
- σ is a constant volatility parameter,
- W is a standard Brownian motion,
- θ is chosen to match the initial term structure of risk-free rates.

We discretize the short-rate process above based on a trinomial tree (see [Canale, 2000] for a description of the tree building technique). At an exercise time slice, the intrinsic (European) option payoff has the form

$$\max\left(\beta_{t'} \sum_{i=1}^{m-1} E^Q\left(\frac{r_i}{\beta_{\tau_{i+1}}} \middle| F_{t'} \right) - \beta_{t'} \sum_{i=1}^{n-1} E^Q\left(\frac{p_i}{\beta_{t_{i+1}}} \middle| F_{t'} \right), \ 0 \right)$$

where

- $\beta_t = e^{\int_0^t r(s)ds}$ is the risk-neutral numeraire value to time t,
- E^Q denotes risk-neutral expectation.

Consider the swap's payer leg, which is specified with reset points, $t_1, \ldots, t_M$, as described in Section 2. Furthermore assume that an interest rate based amount, $R(t_{i+1} - t_i)$, is payable at time t_{i+1}, for $i = 1, \ldots, M-1$, where

$$R = \frac{1 - P(T_p, T_{p+fm})}{\sum_k \left(\sum_l \Delta_{p+f(k-1)+l-1} \right) P(T_p, T_{p+fk})}$$

is an m-period CMS rate with frequency, f. At each reset point t_i, for $i = 1, \ldots, M-1$, we consider an at-the-money, European payer swaption with payoff at t_i of the form

$$\max(R - R_0, 0) \sum_k \left(\sum_l \Delta_{p+f(k-1)+l-1} \right) P(T_p, T_{p+fk}) \qquad (4.1.1.1)$$

where

$$R_0 = \frac{P(0, T_p) - P(0, T_{p+fm})}{\sum_k \left(\sum_l \Delta_{p+f(k-1)+l-1} \right) P(0, T_{p+fk})}$$

is the forward value of R at time zero.

In order to calibrate the Jamshidian Libor rate model parameters, (3.1.2), we include, in the calibration set of instruments, the European payer swaption with payoff (4.1.1.1), for $i = 1, \ldots, M-1$, with corresponding price calculated from our benchmark short-rate trinomial tree model.

To calculate the price of a European style option with payoff at maturity of the form (3.2.1), we directly simulate the Libor-rate process, $\vec{L}(t_i)$, for $i = 1, \ldots, N-1$, by discretizing the SDE (3.1.1) based on an explicit Euler scheme. We seek to evaluate

$$E\left(\max\left(N_{t'} \sum_{i=1}^{m-1} E\left(\frac{r_i}{N_{\tau_{i+1}}} \middle| F_{t'} \right) - N_{t'} \sum_{i=1}^{n-1} E\left(\frac{p_i}{N_{t_{i+1}}} \middle| F_{t'} \right),\ 0 \right) \middle/ N_{t'} \right) \qquad (4.2.1)$$

Our valuation method involves nested, outer and inner, Monte Carlo simulation loops.

For the outer loop, we simulate the Libor rate process to the maturity time, t'. From this point, we value the conditional expectation terms in (4.2.1) using an inner Monte Carlo loop. We then average the sample payoffs over the number of outer loop sample paths.

Here we employed the Numerical Recipes in C routine, *ran2()*, in conjunction with *gasdev()*.