Pricing Inflation Swap, Cap and Floor

A model is presented for pricing swaps, caps, and floors on inflation index returns. To capture general term structures of interest rates and index volatilities, the model requires time-averaged forward rate, and volatility inputs.

For a series of reset times

$$T_0 \leq 0 < T_1 < \cdots < T_N = T,$$

we consider swap-type payments of the following form

- At maturity time T:
  - $\dfrac{I_T}{I_{T_0}} - 1,$ (Zero swap)
  - $\max\left[0, \left(\dfrac{I_T}{I_{T_0}} - 1\right) - X\right],$ (Zero cap)
  - $\max\left(0, X - \left(\dfrac{I_T}{I_0} - 1\right)\right).$ (Zero floor)
- At each reset time T_k:
  - $\dfrac{I_{T_k}}{I_{T_{k-1}}} - 1,$ (Reset swap)

- $\max\left[0,\left(\frac{I_{T_k}}{I_{T_{k-1}}}-1\right)-X\right],$ (Reset cap)
- $\max\left(0,X-\left(\frac{I_{T_k}}{I_{T_{k-1}}}-1\right)\right).$ (Reset floor)

where

- I_t denotes the level of the inflation index at time t,
- X denotes the reset cap (floor) return strike.

We assume that the level of the inflation index satisfies a domestic risk-neutral SDE of the form

$$\frac{dI_t}{I_t}=\left(r(t)-\delta(t)\right)dt+\sigma(t)dB_t \qquad (1)$$

where

- r is the deterministic domestic continuously compounded risk-free,
- δ is a deterministic continuous dividend yield parameter,
- σ is a deterministic volatility parameter,
- B_t is a standard Brownian motion.

Given SDE (1) for index levels, we may model

$$\frac{I_T}{I_t}=F\left(0,t,T\right)\exp\left[-\frac{\left(\sigma_{t,T}\right)^2\left(T-t\right)}{2}+\sigma_{t,T}\left(W_T-W_t\right)\right], \qquad (2)$$

where

- $F\left(0,t,T\right)=e^{\int_t^T\left(r(s)-\delta(s)\right)ds},$

- $\sigma_{t,T} = \sqrt{\frac{1}{T-t}\int_t^T \sigma^2(s)\,ds}$,
- W_t is a standard Brownian motion.

The model

$$\frac{I_T}{I_t} = F(0,t,T)\exp\left[-\frac{\left(\sigma_{t,T}^{WM}\right)^2 T}{2} + \sigma_{t,T}^{WM} W_T^{WM}\right], \tag{3}$$

where

- $F(0,t,T) = e^{\int_t^T (r(s)-\delta(s))ds}$,
- $\sigma_{t,T}^{WM} = \frac{1}{\sqrt{T}}\sqrt{\int_t^T \sigma^2(s)\,ds}$,
- W_t^{WM} is a standard Brownian motion.

We note that, in marginal distribution, Equations 2 and 3 are equivalent.

In the inflation index market, forward levels $F(0,0,\tau_i)$ for a series of "key maturities" $\{\tau_i\}_{i=1}^m$ are traded. In particular, forward levels of the form

$$E_0^{\tau}\{I_\tau\} = \frac{1}{Z_0(\tau)} E_0^Q\left\{e^{-\int_0^\tau r_s ds} I_\tau\right\},$$

where

- $Z_0(\tau)$ is the domestic discount factor, at time 0, to time τ,

- r is the domestic continuously compounded risk-free rate.

Given the assumptions described in Sections 3.1 and 3.2 above, we have

$$E_0^{\tau}\left\{I_{\tau}\right\} = I_0 e^{\int_0^{\tau}(r(s)-\delta(s))ds}$$

and

$$E_0^{Q}\left\{\frac{I_{\tau+\Delta}}{I_{\tau}}-1\right\} = e^{\int_{\tau}^{\tau+\Delta}(r(s)-\delta(s))ds}-1 = \frac{F\left(0,0,\tau+\Delta\right)}{F\left(0,0,\tau\right)}-1 \qquad (4)$$

for all $\tau,\Delta > 0$.

References:

https://finpricing.com/lib/FiBond.html