Pricing American Swaption

A model is presented for pricing single-currency, American style fixed-for-floating interest rate swaptions

If an American swaption is exercised at a point that is not a reset date, in practice, the effective Libor rate at the point of exercise is a blended rate, which is linearly interpolated from a pair of Libor rates with respective accrual periods that bracket the remaining time interval to the next reset date. The effective Libor rate at the exercise point is taken to be the simple interest rate implied from the zero-coupon bond price to the next reset date. This treatment represents a compromise between accuracy and computational efficiency, since it avoids having to determine bracketing Libor rate values.

We consider a single currency swap specified as follows,

- reset point, T_i, for $i = 0,\ldots,M$, where $0 < T_0 < \ldots < T_M$,
- floating-leg payment, $N_i \, L(T_i;T_i,T_{i+1}) \, \Delta_i$, at settlement time, T_{i+1} , for $i = 0,\ldots,M-1$, where
    - N_i is a notional amount,
    - $\Delta_i = T_{i+1} - T_i$,
    - $L(T_i;T_i,T_{i+1}) = \dfrac{1}{\Delta_i}\left(\dfrac{1}{P(T_i,T_{i+1})} - 1\right)$ is the simple interest rate[1] applicable at T_i for the accrual period, Δ_i ,

[1] Here $P(T_i,T_{i+1})$ denotes the price at T_i of a zero-coupon bond with maturity, T_{i+1}, and unit face value.

- fixed-leg payment, $N_i\, R_i\, \Delta_i$, at settlement time, T_{i+1} , for $i = 0,\dots,M-1$, where R_i is a simple, annualized rate (ref https://finpricing.com/lib/IrCurveIntroduction.html).

An American style swaption allows the holder to choose the entry point, into the tail of the swap, from a list of possible exercise times (e.g., a window of successive business days). Specifically let t_i and τ_i, for $i = 1,\dots,n$, where $t_i \le \tau_i$,

$$0 < t_1 < \dots < t_n ,$$

and

$$T_0 \le \tau_1 < \dots < \tau_n \le T_M ,$$

denote a respective notification time and exercise time. Next consider a particular such pair of times, t and τ, and assume that

$$T_i < \tau < T_{i+1}$$

for some $i \in \{0,\dots, M-1\}$. If notification is given at time t, then the respective floating rate and fixed rate payments,

$$N_j\, L(T_j; T_j, T_{j+1})\, \Delta_j$$

and

$$N_j\, R_j\, \Delta_j ,$$

must be made at T_{j+1}, for $j = i+1,\dots,M-1$. In addition a blended Libor rate, $\hat{L}$, is determined at τ, and the respective floating rate and fixed rate payments,

$$N_i\, \hat{L}\left(T_{i+1} - \tau\right)$$

and

$$N_i \; R_i \left(T_{i+1} - \tau\right) \; ,$$

must be made at time T_{i+1}.

We consider the “*BK*” method for valuing single-currency, fixed-for-floating interest rate, American style swaptions with features of the type described in Section 2. The *BK* method is an implementation of a “disconnected” tree discretization of a one factor Black-Karazinski (BK) risk-neutral short-rate process of the form below.

Let r denote the short-interest rate. We consider a short-interest rate process such that $\log r$ satisfies a risk-neutral SDE of the form,

$$d \log r_t = \left(\theta_t - a_t \log r_t\right) dt + \sigma_t dW_t \; , \tag{3.1}$$

where

- a_t is a piecewise constant mean reversion rate,
- σ_t is a piecewise constant volatility function,
- θ_t is chosen to fit the initial term structure of discount factors,
- W_t is a standard Brownian motion.

A disconnected tree discretization of the short-rate process above is non-recombinant by design, but employs an interpolation scheme to approximate short-rate values at tree nodes along a time slice.

Calibration is accomplished by matching, in a least squares sense, the model price against the market price for each respective European style payer swaption in a cache of calibration securities. The volatility break points are related to the forward start times of the respective swaptions in the calibration portfolio (see Section 4.2 for a typical specification). Given an American swaption, Consider the American swaption. Let r_t denote the short-interest rate at time t. We assume that the process, $\{\log r_t \mid 0 < t \le T_M\}$, satisfies a risk-neutral SDE of the Black-Karazinski form, (3.1).

We construct a trinomial tree to approximate the short-rate process, $\{r_t \mid 0 < t \le T_M\}$, based on the algorithm. Let

$$\Omega = \{t_i\}_{i=0}^{m},$$

where $0 = t_0 < ... < t_m = T_M$, be a partition of the interval, $[0, T_M]$, from valuation time to the last settlement point. We assume that the respective volatility and mean reversion functions, σ_t and a_t, are constant over each interval, $[t_{i-1}, t_i)$, for $i = 1, \ldots, m$. Moreover, we ensure that the set of time slices, Ω, includes the following events,

- the reset point, T_i, for $i = 0, \ldots, M$,
- the respective exercise point and notification point, τ_i and t_i, for $i = 1, \ldots, n$
- the respective sets of volatility time and mean reversion time break points.

Consider a particular exercise point, τ, and assume that

$$T_{i_\tau} \le \tau < T_{i_\tau + 1}$$

where $i_\tau \in \{0, ..., M-1\}$; furthermore, let t be the corresponding notification time. Consider an option to enter, at τ, into the tail of the swap above with right to pay fixed and receive float. Based on Section 2 specifications and the treatment in Optex of the effective Libor rate value at exercise time, this swap has price at notification time, t,

$$\Phi_t = \max\left[E^Q\left(\frac{N_{i_\tau}\left[L(\tau;\tau,T_{i_\tau+1}) - R_{i_\tau}\right](T_{i_\tau+1}-\tau)}{e^{\int_t^{T_{i_\tau+1}} r(s)ds}} + \sum_{k=i_\tau+1}^{M-1} \frac{N_k\left[L(T_k;T_k,T_{k+1}) - R_k\right]\Delta_k}{e^{\int_t^{T_{k+1}} r(s)ds}} \middle| F_t \right), 0 \right],$$

$$= \max\left(N_{i_\tau}\left[P(t,\tau) - \left(1+\left[T_{i_\tau+1}-\tau\right]R_{i_\tau}\right)P(t,T_{i_\tau+1})\right] + \left(\sum_{k=i_\tau+1}^{M-1} N_k\left[P(t,T_k) - (1+R_k\Delta_k)P(t,T_{k+1})\right]\right), 0\right),$$

where E^Q denotes the risk-neutral probability measure. The American swaption value is then given by

$$\Omega = \max_{t\in\{t_i\}_{i=1}^n} E^Q\left(\frac{\Phi_t}{e^{\int_0^t r(s)ds}}\right)$$

where t denotes a stopping time.

Consider the Bermudan swaption specified in Section 2. Let

$$\Lambda = \{T_{i_j}\}_{j=1}^p,$$

where $T_{i_1} < \ldots < T_{i_p}$, be a subset of the set of reset points, $\{T_i\}_{i=0}^M$. We consider a portfolio of p European style payer swaptions. In particular, the j^{th} ($j=1,\ldots,p$) swaption is specified as follows,

- forward start time, T_{i_j},
- resets, $T_{i_j},\ldots,T_{M-1}$,
- payoff at T_{i_j}, $\max\left(1-\left(X\sum_{k=i_j}^{M-1}\Delta_k P(T_{i_j},T_{k+1}) + P(T_{i_j},T_M)\right),0\right)$, where X is a strike level.

For each option in the portfolio above, we obtain from WM a corresponding implied Black's volatility. We then price the payer swaption above based on Black's analytical formula. Let P_i denote the price for the i^{th} ($i = 1,\ldots,p$) swaption calculated using Black's model as described above.

We assume that the volatility function, σ_t, is constant over the respective periods, $[0, T_{i_1})$, $[T_{i_1}, T_{i_2}),\ldots[T_{i_{p-1}}, T_M)$. Let σ_i, for $i = 1,\ldots,p$, be the constant volatility value, which corresponds to the respective intervals $[0,\tau_1)$, $[\tau_1,\tau_2),\ldots[\tau_{p-1},T_M)$. Furthermore let $bench(\vec{\sigma};i)$, for $i = 1,\ldots,p$, denote the benchmark option price for the i^{th} swaption where $\vec{\sigma} = \left[\sigma_1,\ldots,\sigma_p\right]^T$. We seek to solve in a least squares sense,

$$\vec{F}(\vec{\sigma}) = \begin{bmatrix} bench(\vec{\sigma};1) - P_1 \\ \vdots \\ bench(\vec{\sigma};l) - P_l \end{bmatrix} = \vec{0},$$

for the unknown, $\vec{\sigma}$; that is, we seek to minimize

$$\vec{F}(\vec{\sigma})^T \vec{F}(\vec{\sigma}) = 0$$

with respect to $\vec{\sigma}$. A necessary condition for a minimum is that

$$\left[\frac{\partial}{\partial\vec{\sigma}} \vec{F}(\vec{\sigma})^T\right] \vec{F}(\vec{\sigma}) = \vec{0} \in R^p. \qquad (4.2.1)$$

We solve (4.2.1) for the unknown, $\vec{\sigma}$, using Newton's method based on our short-rate trinomial tree.