Multi-currency BGM Pricing Model

The Brace-Gatarek-Musiela (BGM) model is a multi-factor log-normal model. This model applies to both currencies. Its principle is to fix a tenor d, for instance 3 months, and to assume that each Libor rate at date t, $L(t,d)$ has a log-normal distribution in the "forward-neutral" probability of maturity $t+d$. The present model uses 4 factors, which we may assume independent.

Let $F(t,T,d)$ be the FRA from T to $T+d$ as observed at date t. The diffusion of $F(t,T,d)$ with respect to *t* is given, as in the Heath-Jarrow-Morton (HJM) model, by a combination of factors:

$$dF(t,T,d) = F(t,T,d)\left(\mu(t,T)\,dt + \sum_{i=1}^{4} \varphi_i(t,T) \;\; dW_i \right)$$

where the Brownian motions W_i are independent and the drift μ is adapted to make the model arbitrage free. The factors $\varphi_i(t,T)$ are defined by their "relative" size with respect to $\varphi_1(t,T)$ and by a "volatility triangle":

$$\varphi_i(t,T) = \psi_i(T-t)\,\sigma(t,T)$$

The "relative" factors ψ_i are parameterized in the following way:

$$\psi_1(t) = c \qquad \psi_2(t) = \sqrt{\frac{1-c^2}{1+\left(t/a\right)^{\alpha}}}$$

$$\psi_3(t) = \sqrt{\left(1-c^2\right)\left(1-\frac{1}{1+\left(t/b\right)^{\beta}}\right)}$$

$$\psi_4(t) = \sqrt{1-\psi_1(t)^2-\psi_2(t)^2-\psi_3(t)^2}$$

$$= \sqrt{\left(1-c^2\right)\left(\frac{1}{1+\left(t/b\right)^{\beta}}-\frac{1}{1+\left(t/a\right)^{\alpha}}\right)}$$

The sum of the squares of all the factors is imposed to be identically 1 so that the local volatility of each single Libor future rate is precisely $\sigma(t,T)$. The parameters a, b, c, α and β are input in the HJM spreadsheet under the following names:

$c =$ shift $\qquad a =$ sector 1 $\qquad b =$ sector 2

$\alpha =$ slope 1 $\qquad \beta =$ slope 2

The volatility triangle $\sigma(t,T)$ bears this name because it is defined for $t \le T$. It is discretized (see next sect.) and its values on the different cells are calibrated to match the market price of caplets and swaptions, except for diagonal values ($t = T$) which are input by the user (the so-called “s-vols”).

A series of dates $T_1 < \cdots < T_n$, spaced approximately by the tenor d, is fixed, for instance the IMM maturity dates. At each date t, the only available information is the short rate $r(t)$, the spot Libor rate $L(t) = F(t,t,d)$, the series of future rates $F_i(t) = F(t,T_i,d)$, $i = i_1(t),\ldots,n$ where $i_1(t)$ is the first index such that $T_i > t$, and the stub rate $s(t)$, which applies on the period $\lfloor t, T_{i_1(t)} \rfloor$. In practice, the time t is discretized and the short rate is that which applies over the time period of a diffusion step.

Discount factors are computed from the rates as follows:

$$DF(t,T_{i_1(t)}) = \frac{1}{1+(T_{i_1(t)} - t)\,s(t)}$$

$$DF(t,T_{i+1}) = \frac{DF(T_i)}{1+(T_{i+1} - T_i)\,F_i(t)} \qquad i = i_1(t),\ldots,n-1$$

Option values are discounted risk-neutral expectations of their pay-off. In a stochastic interest rate environment, the discounting should be taken as the accumulation of the spot rate $r(t)$. We define the *numeraire* at date t actualized in 0 as:

$$Num(t) = \exp\left(-\int_0^t r(u)\,du\right)$$

Again, a discretized version of this formula is used in practice (see sect. II).

The arbitrage theory states that:

$$\mathbf{E}_t\left(\frac{Num(T)}{Num(t)}\right) = DF(t,T)$$

The Bayesian rule implies:

$$\mathbf{E}_0\left(Num(t)\ DF(t,T)\right) = DF(0,T)$$

The above structure applies to both currencies. In the sequel, superscripts d and f will specify whether we refer to domestic or foreign rates.

We denote by $X(t)$ the exchange rate at date t. It follows a diffusion process defined by:

$$\frac{dX}{X} = \left(r^d - r^f\right)dt + \sigma_X(t)\,dW_X$$

Short rates r^d and r^f are processes described in the previous section.

The volatility σ_X can either be a time dependent parameter (deterministic volatility) or itself a process (stochastic volatility). In the latter case, its diffusion equation is:

$$d\sigma_X = \mu(t)\left(\overline{\sigma}(t) - \sigma_X\right)dt + \eta(t)\sigma_X\ dW_\sigma$$

This process is positive. As indicated, parameters μ, $\overline{\sigma}$ and η can be time dependent. The first one is the *Mean Reversion,* the second one is the *Expected Spot Volatility* and the third one is the *Volatility of Volatility* or *Vvol.* The two Brownian motions W_X and W_σ can be correlated, with a time dependent correlation $\rho^{X\sigma}(t)$. Parameters μ, $\overline{\sigma}$ and η must be positive, but $\rho^{X\sigma}$ can have either sign.

In terms of volatility surface, the Vvol introduces a positive smile, the correlation induces a skew and the mean reversion makes the smile decrease with maturity. The expected spot volatility drives the term structure.

As mentioned previously, Brownian motions $W_1^d, \ldots, W_4^d$ are uncorrelated, as well as $W_1^f, \ldots, W_4^f$. However, we allow W_i^d and W_i^f to be correlated with a possibly time dependent correlation $\rho_i^{df}(t)$. In order to avoid almost useless complexity, we assume that W_i^d and W_j^f are not correlated for non-equal indices i and j.

The Brownian motion W_X can be correlated to interest rate ones, with possibly time dependent correlations $\rho_1^{Xd}(t), \ldots, \rho_4^{Xd}(t)$, $\rho_1^{Xf}(t), \ldots, \rho_4^{Xf}(t)$. In order to avoid non-positive definite correlation matrices, the correlation of W_σ with $W_i^{d \text{ or } f}$ is set to the product $\rho^{X\sigma}(t)\,\rho_i^{Xd \text{ or } Xf}(t)$. If we decompose W_σ into a component totally correlated to W_X and another independent, then this is the same thing as saying that the independent component is also independent of the interest rate Brownian motions (ref https://finpricing.com/lib/IrInflationCurve.html).

There is one theoretical subtlety about multi-currency models. Risk-neutral probabilities differ in both currencies, because numeraires are different. In the domestic risk-neutral probability, the expectation of the daily discounted value of a unit of domestic currency is equal to the domestic discount factor:

$$\mathbf{E}^d\left(Num^d(t)\right) = DF^d(0,t)$$

The same applies to a unit of foreign currency, and this yields:

$$\mathbf{E}^d\left(Num^d(t)\ X(t)\right) = X(0)\ DF^f(0,t)$$

By the Bayesian rule, we get:

$$\mathbf{E}^d\left(Num^d(t)\ X(t)\,DF^f(t,T)\right) = X(0)\ DF^f(0,T)$$

In the foreign risk-neutral expectation, one would have:

$$\mathbf{E}^f\left(Num^f(t)\right) = DF^f(0,t)$$

If domestic rates and FX are correlated, this shows that expectations $\mathbf{E}^d$ and $\mathbf{E}^f$ cannot coincide. Similarly, the non-discounted expectation of the exchange rate is not equal to the forward exchange rate:

$$\mathbf{E}^d\left(X(t)\right) \neq \frac{\mathbf{E}^d\left(Num^d(t)\ X(t)\right)}{\mathbf{E}^d\left(Num^d(t)\right)} = X(0)\ \frac{DF^f(0,t)}{DF^d(0,t)}$$

The price of an option is also the risk-neutral expectation of its discounted pay-off. Consequently, if the pay-off is set in domestic currency, the price of the option in domestic currency is:

$$P^d = \mathbf{E}^d\left(Num^d(T)\, PayOff^{\,d}(T)\right)$$

However, if it is set in foreign currency, then the price of the option in domestic currency is:

$$P^d = \mathbf{E}^d\left(Num^d(T)\, X(T)\, PayOff^{\,f}(T)\right)$$

and, in foreign currency:

$$P^f = \frac{1}{X(0)}\, \mathbf{E}^d\left(Num^d(T)\, X(T)\, PayOff^{\,f}(T)\right)$$

In the forward risk-neutral probability, one would have:

$$P^f = \mathbf{E}^f\left(Num^f(T)\, PayOff^{\,f}(T)\right)$$

A discretized version of these formulas will be used in the next section to compute option prices and sensitivities.