Pricing Seller Swap

One party sells mortgage pools on its balance sheet and pays the bond interest by entering into a pay-fixed swap with CHT, and receives the interest from MBS pool sold to CHT. This is a seller swap. The fixed leg is semi annual, and the float leg, MBS coupons, is monthly. In addition, the MBS sold to the trust generates principal cash flows. CHT buys new-pooled mortgages from the party with this principal flows every month until the maturity of the swap.

CHT sold bullet bonds of total notional N that pays C% semi annual coupon for 60 months to investors for P dollars. Using these proceedings, CHT bought a mortgage pool, whose dollar price was P, from the party and entered into a "seller swap" with the party. CHT will buy new mortgage pools from the principal payments generated from the mortgage pools that already bought from the party

The party agreed to pay CHT the coupon of the bond, $N \cdot C / 200$ dollars every six months for 5 years. On the floating side, CHT agreed to pay, monthly, all the coupons generated by the mortgage pools it bought from the party for 60 months.

Let t_i denotes the first day of month *i*, for $i = 1, \ldots 61$, assuming t_1 is the inception of the swap and C_i denotes the coupon the party receive at time t_{i+1} for $i = 1, \ldots 60$. Furthermore, let p_j^j and α_j denote the time t_j dollar price and unit price of one dollar of notional, respectively of the mortgage pool, MBS^j, sold to CHT at time t_j. Moreover, assume the valuation date *t* belongs to $[t_m, t_{m+1})$, where *m* is an integer between 1 and 60. Then

The fair value of the Seller-Swap

$$= \sum_{k=1}^{m} \text{time } t \text{ dollar price of } MBS^k - \text{ time } t \text{ dollar price of the bond}$$

$$-\left(\frac{\beta_1}{\alpha_1}-1\right)N\,d(t,t_{61}) - \sum_{j=2}^{m} p_j^j\left(\frac{1}{\alpha_j}-1\right)d(t,t_{61}) - E\left(e^{-\int_t^{t_{61}} r(s)ds}\sum_{j=m+1}^{60} p_j^j\left(\frac{1}{\alpha_j}-1\right)\right),$$

where $d(t,t_{61})$ is time *t* price of a zero coupon bond that matures at time t_i and β_1 is the price of one dollar notional of the bond at inception.

The time *t* price of the fixed leg, which includes the notional at maturity, is time *t* price of the bond less present value of the notional, *N*.

We show that the time *t* value of the float- leg is equal to the sum of time *t* dollar prices of the mortgages the party sold to CHT less expected discounted value of mortgages at the maturity of the swap.

Let P_i and N_i denotes the time t_i price and notional of all the mortgages the party sold to CHT up to and including time t_i for $i = 1,..60$ respectively.

At inception, t_1, the party sold one MBS, MBS^1, to CHT.

Evolution of MBS^1

time t_1 dollar Price of $MBS^1 : p_1^1$, measurable at time t_1

Coupon for the first month: b_1^1, measurable at time t_1

Then $C_1 = b_1^1$ and $P = p_1^1 = P_1$.

Principal payment at time t_2 : q_2^1, measurable at time t_2

time t_2 dollar Price of MBS^1 : p_2^1, measurable at time t_2

Coupon for the second month: b_2^1, measurable at time t_2

At time t_2 the party received q_2^1 and sold another MBS, MBS^2, to CHT. i.e.,

$$q_2^1 = p_2^2.$$

where p_2^2 is the time t_2 price of MBS^2. Additionally, we note that

$$P_2 = p_2^1 + p_2^2 \text{ and } N_2 = N_1 - p_2^2 + \frac{p_2^2}{\alpha_2},$$

where α_2 is the time t_2 price of MBS^2 per one dollar of notional.

Coupon C_i received at time t_{i+1} is given by

$$C_i = \sum_{k=1}^{i} b_i^k. \qquad (1)$$

The new mortgage pool, MBS^i , created from the sum of the prepayments, $\sum_{k=1}^{i-1} q_i^k$. Therefore,

$$\sum_{k=1}^{i-1} q_i^k = p_i^i. \qquad (2)$$

Furthermore, we note

$$P_i = \sum_{k=1}^{i} p_i^k \quad \text{and} \quad N_i = N_{i-1} - p_i^i + \frac{p_i^i}{\alpha_i}, \qquad (3)$$

where α_i is the time t_i price of MBS^i per one dollar of notional.

Since, price of an MBS at the beginning of a month is the expected discounted values of coupon, prepayment and the price of the MBS at the end of the month, we observe the following set of equations.

$$p_i^j = E\left(e^{-\int_{t_i}^{t_{i+1}} r(s)ds} \left(b_i^j + q_{i+1}^j + p_{i+1}^j \right) \middle| \zeta_i \right) \quad \text{for } j = 1,2,\ldots,i, \qquad (4)$$

where

- E is the risk neutral expectation,
- r is the risk free short rate (see https://finpricing.com/lib/IrCurve.html), and
- ζ_i is the filtration at time t_i.

Equations (1), (2), (3) and (4) imply

$$P_i = E\left(e^{-\int_{t_i}^{t_{i+1}} r(s)ds} (C_i + P_{i+1}) \middle| \zeta_i \right) \quad (5)$$

We assume that the length of the swap is 60 months. At the end of month 60, CIBC will receive, $C_{60} = \left(\sum_{k=1}^{60} b_{60}^k \right)$, coupons from 60 mortgages. Let P_{61} denote the value of these 60 mortgages and their prepayments on the last day of month 60. Then

$$P_{61} = \sum_{k=1}^{60} \left(p_{61}^k + q_{61}^k \right).$$

Let F_t denotes the value of the floating cash flows on the valuation date. Then,

$$F_t = I_m + ... + I_i + ... + I_{59} + I_{60}, \quad (6)$$

where,

$$I_j = E\left(e^{-\int_{t}^{t_{j+1}} r(s)ds} C_j \right) \text{ for } j = m, m+1, ..., 60.$$

First, we work on the last expectation, I_{60}, in Equation (6).

$$I_{60} = E\left(e^{-\int_t^{t_{61}} r(s)ds} C_{60} \right) = E\left(e^{-\int_t^{t_{60}} r(s)ds} E\left(e^{-\int_{t_{60}}^{t_{61}} r(s)ds} \left(C_{60} + P_{61}\right) \middle| \zeta_{60} \right)\right) - R, \tag{7}$$

where R is given by

$$R = E\left(e^{-\int_t^{t_{61}} r(s)ds} P_{61} \right). \tag{8}$$

Applying Equation (5) with appropriate indices to Equation (7) yields

$$I_{60} = E\left(e^{-\int_t^{t_{60}} r(s)ds} P_{60} \right) - R.$$

The sum of the last two terms in Equation (6), $I_{59} + I_{60}$, is

$$I_{59} + I_{60} = E\left(e^{-\int_t^{t_{60}} r(s)ds} \left(C_{59} + P_{60}\right) \right) - R$$

$$= E\left(e^{-\int_t^{t_{59}} r(s)ds} E\left(e^{-\int_{t_{59}}^{t_{60}} r(s)ds} \left(C_{59} + P_{60}\right) \middle| \xi_{59} \right)\right) - R$$

Again, using Equation (5) with appropriate indices, we obtain

$$I_{59} + I_{60} = E\left(e^{-\int_t^{t_{59}} r(s)ds} P_{59} \right) - R$$

Similarly we can show that

$$I_{58} + I_{59} + I_{60} = E\left(e^{-\int_t^{t_{58}} r(s)ds} P_{58} \right) - R$$

By continuing this backward addition process of terms in Equation (6), we arrive at

$$\begin{aligned} F_t &= P_t - R \\ &= \sum_{k=1}^{m} \left(\text{time t dollar price of MBS}^{\text{k}} \right) - R \end{aligned} \tag{9}$$