# Lookback Call Option Model

A model is presented for pricing a European lookback call option on a stock index with guaranteed exchange rate (LBCGER).

The LBCGER specification includes an exercise time, $T$ (where $T > 0$), the guaranteed exchange rate, *GER*, and two parameters, $look_{\min}$ and $look_{\max}$ (where $0 \le look_{\min} < look_{\max} \le T$), which define a lookback window $[look_{\min}, look_{\max}]$. In addition a sampling frequency (e.g., daily, weekly, etc.) over the lookback window is specified.

Let $S_t$ denote the value at time $t$ for the underlying security. The strike of the LBCGER, $K$, is set equal to the minimum price of the underlying security over a set of discrete points, $\{t_i\}_{i=1}^{N}$, which partition the lookback window according to the sampling frequency. That is,

$$K = \min_{i=1,\dots,N} S_{t_i}$$

where $look_{\min} = t_1 < \cdots < t_N = look_{\max}$. If, for example, the lookback window is to be partitioned into $N$ uniformly spaced points, then,

$$t_i = look_{\min} + (i-1)\Delta t\,,$$

for $i = 1,\ldots,N$, where $\Delta t = \dfrac{look_{\max} - look_{\min}}{N-1}$.

The payoff at maturity is the value of a standard European call with strike $K$ adjusted by the guaranteed exchange rate $GER$, that is,

$$\begin{cases} GER \times (S_T - K), & \text{if } S_T > K, \\ 0, & \text{otherwise.} \end{cases}$$

The method for pricing a lookback call option with guaranteed exchange rate is based on a single factor Monte Carlo approach. The idea of the method is to stochastically generate a large number of discrete sample paths for the underlying security.

For each path, the minimum value of the underlying security over the set of lookback window sample times, $\{t_i\}_{i=1}^{N}$, is recorded and used to compute the quantity adjusted payoff (by a certain application of the Black-Scholes pricing formula). The payoffs for each path are then combined to provide an expected payoff for the option. Next we describe the method in detail.

Risk neutral pricing formulas are presented for various types of cross-currency instruments, in particular, European call options with payoffs at maturity of the form

$$\max(0,[S_T - K] \times GER).$$

Let $X_t$ and $S_t$ denote respectively the value of the foreign exchange rate and the price of the underlying security at time $t$. According to Wei, the processes $\{S_t | t \in [0,T]\}$ and $\{X_t | t \in [0,T]\}$ respectively satisfy, under the (domestic) risk neutral probability measure, the SDEs

$$\begin{cases} \frac{dS_t}{S_t} = (r_f - q - \rho\sigma_X\sigma_S)dt + \sigma_S dW_t, & t \in [0,T], \quad (3.1.1a) \\ \frac{dX_t}{X_t} = (r_d - r_f)dt + \sigma_X dZ_t, & t \in [0,T], \quad (3.1.1b) \end{cases}$$

where $\{Z_t | t \in [0,T]\}$ and $\{W_t | t \in [0,T]\}$ are standard Brownian motions with constant instantaneous correlation $\rho$ (here $r_d, r_f, q, \sigma_S$ and $\sigma_X$ are certain constants described below). In (3.1.1a), $r_f$ denotes the foreign risk-free rate for the time period $[0,T]$, $q$ is the continuous dividend yield for $S_t$ over the period $[0,T]$, and $\sigma_S$ is the instantaneous volatility for the proportional change, $\frac{dS_t}{S_t}$, over the period $[0,T]$. In (3.1.1b), $r_d$ represents the domestic risk-free interest rate https://finpricing.com/lib/IrCurve.html) for the period $[0,T]$, and $\sigma_X$ is the instantaneous volatility in the proportional change, $\frac{dX_t}{X_t}$, over $[0,T]$.

In this section we formulate the price of the LBCGER at time equal to zero. Let $\{t_i\}_{i=1}^{N}$ be a partition of the lookback window $[look_{\min}, look_{\max}]$. Recall, from Section 2, that

$$f(S_T, K) = \begin{cases} GER \times (S_T - K), & \text{if } S_T > K, \\ 0, & \text{otherwise,} \end{cases}$$

is the payoff at maturity for the LBCGER (here $S_T$ is the price for the underlying security at maturity and $K = \min_{i=1,\dots,N} S_{t_i}$ is the minimum of the underlying security price over the discrete set of sample times $\{t_i\}_{i=1}^{N}$). At time equal to zero, the price of the LBCGER is equal to the discounted expected payoff at maturity,

$$e^{-r_d T} E(f(S_T, K)). \qquad (3.2.1)$$

However, by the law of iterated conditional expectations, (3.2.1) is equal to

$$e^{-r_d T} E[E(f(S_T, K) | F_{t_N})] \qquad (3.2.2)$$

(here $\{F_t | t \in [0, T]\}$ is the filtration induced by the process $\{W_t | t \in [0, T]\}$). The formulation (3.2.2) for the price at time zero has certain computational advantages (as we will see in Sections 3.3 and 3.4).

Next we show how to approximate (3.2.1).

From (3.2.2) and by algebraic manipulation, the price of the LBCGER at time zero is equal to

$$e^{-r_d t_N} E[e^{-r_d (T-t_N)} E(f(S_T, K)|F_{t_N})] .$$

Notice, however, that the conditional expectation

$$e^{-r_d (T-t_N)} E(f(S_T, K)|F_{t_N}) \qquad (3.3.1)$$

can be viewed as the price of a European call on a domestic asset with dividend yield of

$$\hat{q} = -(r_f - r_d - q - \rho\sigma_S\sigma_X) . \qquad (3.3.2)$$

To be specific, let $BS(\overline{T}, \sigma, \overline{r}, \overline{q}, spot, strike)$ denote the Black-Scholes price of a European call, where $\overline{T}$ denotes the option maturity, $\sigma$ denotes volatility, $\overline{r}$ denotes the riskless interest rate, $\overline{q}$ denotes the continuous dividend yield, $spot$ is the initial value for the underlying security, and $strike$ is the option strike level. Then (3.3.1) is equal to

$$BS(T - t_N, \sigma_S, r_d, \hat{q}, S_{t_N}, K) \times GER .$$

Next we describe a Monte Carlo technique, based on the Black-Scholes analysis above, for computing the price, (3.2.2), of the LBCGER at time zero.

From the SDE (3.1.1a) and by Ito's lemma, the process $\{S_t | t \in [0, T]\}$ satisfies the SDE

$$d\log S_t = (\mu - \frac{\sigma_S^2}{2})dt + \sigma_S dW_t, \qquad t \in [0,T], \tag{3.4.1}$$

where $\mu = r_f - q - \rho\sigma_X\sigma_S$ is the drift term in the SDE (3.1.1a). A discrete sample path, $\left\{S_{t_i} \,|\, i = 1,\ldots,N\right\}$, can be generated efficiently by the iterative scheme

$$S_{t_{i+1}} = S_{t_i} \exp\left\{(\mu - \frac{\sigma_S^2}{2})\Delta t_i + \varepsilon_i \sigma_S \sqrt{\Delta t_i}\right\}, \tag{3.4.2}$$

for $i = 1,\ldots,N-1$, where $\varepsilon_i$ is a random sample from the standard normal distribution and $\Delta t_i = t_{i+1} - t_i$. Note that, since the drift and volatility parameters in the SDE (3.4.1) are constant, we can "jump" directly to the start, $t_1$, of the lookback window. That is, in the iterative scheme (3.4.2), we set

$$S_{t_1} = \exp\left\{\log S_0 + (\mu - \frac{\sigma_S^2}{2})t_1 + \varepsilon_0 \sigma_S \sqrt{t_1}\right\}$$

where $\varepsilon_0$ a random sample from the standard normal. Note also that the iterative scheme (3.4.2) is not employed by Financial Products, New York; rather, a computationally less efficient sampling scheme, which is described in Section 4, is used.

Let $\Gamma = \left\{S_{t_i} \,|\, i = 1,\ldots,N\right\}$ be a discrete sample path generated by the scheme (3.4.2), and let

$$K = \min_{i=1,\ldots,N} S_{t_i}$$

be the minimum of all discrete sample times in the lookback window. Then

$$payoff_{path}(\Gamma) = e^{-r_d T_N} BS(T - t_N, \sigma_S, r_d, \hat{q}, S_{t_N}, K) \times GER \qquad (3.4.3)$$

is the payoff for path $\Gamma$. A Monte Carlo approximation to (3.2.1), based on $M$ sample paths, is then given by

$$payoff_{MonteCarlo} = \frac{1}{M} \sum_{i=1}^{M} payoff_{path}(\Gamma_i).$$