Computing Conditional Probability of Hitting Barrier

A model is developed for evaluating the conditional probability of hitting an upper barrier before a lower barrier, and vice versa, for a tied down geometric Brownian motion with drift. The method produces an analytical value for this probability, assuming that the barrier levels are constant and continuously monitored.

The method for evaluating the conditional probability of hitting an upper barrier before a lower one appears to produce the correct analytical value, which is expressed as an infinite series of exponentially decreasing terms. In practice, however, this infinite series will need to be truncated to a finite sum; if more accuracy is required, then more terms in the series should be included.

Let *St* denote the price at time equal to *t* of an underlying security. Furthermore assume that the process {*S t* |*t* Î[0,+¥) } satisfies, under some measure *P* , the stochastic differential equation

$$dS_t = S_t\left(\mu dt + \sigma dW_t\right), \qquad t \in [0,+\infty) .$$

Also let

· *Hu* and *Hd* (where *H H u d* >) respectively denote constant upper and lower barrier levels,

· t 1 = inf { } inf *t* ³ 0|,*S* ³ *H t u* and t { } 2 = inf *t* ³ 0|*S* £ *H t d* respectively denote the first hitting times of the barrier levels *Hu* and *Hd* (here we assume that *H S H d u* < < 0), and

· T (where $T > 0$) denote a length of time.

We consider the conditional probability that the upper barrier level is crossed during the interval [0,T], and for a smaller time than for which the lower barrier level is crossed, given that $S_T = y$, that is,

$$P\left(\tau_1 \le T, \tau_1 < \tau_2 \middle| S_T = y\right).$$

An analytical value for this conditional probability is provided in [Myint, 1997]. The derivation is based, in part, on an application of Theorem 4.2 in [Anderson, 1960] (see page 175), which gives an analytical value for a similar conditional probability but with respect to standard Brownian motion (see https://finpricing.com/lib/FxForwardCurve.html)

We first introduce some notation. Specifically let

$$\tau_1^{f,\gamma} = \inf\left\{t \ge 0 \middle| f(t) \ge \gamma\right\}$$

and

$$\tau_2^{f,\gamma} = \inf\left\{t \ge 0 \middle| f(t) \le \gamma\right\}$$

denote first hitting times, respectively from below and from above, of the constant barrier level γ.

Next let

· $\{W_t\}_{t \in [0,+\infty)}$ denote standard Brownian motion under a probability measure P ,
And

· γ_1 and γ_2 (where $\gamma_1 > 0$ and $\gamma_2 < 0$) respectively denote constant upper and lower barrier levels.

For standard Brownian motion, consider the conditional probability that the upper barrier level is crossed during the interval [0,*T*], and for a smaller time than for which the lower barrier level is crossed, given that *W y T* = , that is,

$$P\left(\tau_1^{W,\gamma_1} \le T, \tau_1^{W,\gamma_1} < \tau_2^{W,\gamma_2} \middle| W_T = y\right).$$

From Theorem 4.2 in [Anderson, 1960] (with d d 1 2 = = 0), this conditional probability is equal to

$$\begin{cases} \displaystyle\sum_{n=1}^{+\infty}\left[e^{\frac{-2}{T}\left(n^2\gamma_1(\gamma_1-y)+(n-1)^2\gamma_2(\gamma_2-y)-n(n-1)\left[\gamma_1(\gamma_2-y)+\gamma_2(\gamma_1-y)\right]\right)} - e^{\frac{-2}{T}\left[n^2\left(\gamma_1(\gamma_1-y)+\gamma_2(\gamma_2-y)\right)-n(n-1)\gamma_1(\gamma_2-y)-n(n+1)\gamma_2(\gamma_1-y)\right]}\right], & \text{if } y \le \gamma_1, \\ \displaystyle 1-\sum_{n=1}^{+\infty}\left[e^{\frac{-2}{T}\left((n-1)^2\gamma_1(\gamma_1-y)+n^2\gamma_2(\gamma_2-y)-n(n-1)\left[\gamma_1(\gamma_2-y)+\gamma_2(\gamma_1-y)\right]\right)} - e^{\frac{-2}{T}\left(n^2\left[\gamma_1(\gamma_2-y)+\gamma_2(\gamma_2-y)\right]-n(n+1)\gamma_1(\gamma_2-y)-n(n-1)\gamma_2(\gamma_1-y)]\right)}\right], & \text{if } y \ge \gamma_1. \end{cases}$$

Also for standard Brownian motion consider the probability that the upper barrier level is crossed during the interval [0,*T*], and for a smaller time than for which the lower barrier level is crossed, and that *WT* lies in an interval *I* , that is,

$$P\left(\tau_1^{W,\gamma_1} \le T, \tau_1^{W,\gamma_1} < \tau_2^{W,\gamma_2}, W_T \in I\right).$$

From Bayes' Theorem and (1), this probability is equal to

$$\int_I g(y) P\left(\tau_1^{W,\gamma_1} \le T, \tau_1^{W,\gamma_1} < \tau_2^{W,\gamma_2} | W_T = y\right) dy$$

For the process {*S t* } *t* | Î[0,+¥) , consider the conditional probability that the lower barrier level is crossed during the interval [0,*T*], and for a smaller time than for which the upper barrier level is crossed, given that *S y T* = , that is,

$$P\left(\tau_2^{S,\gamma_2} \le T, \tau_2^{S,\gamma_2} < \tau_1^{S,\gamma_1} | S_T = I\right).$$

FP chooses to obtain this conditional probability by considering the identity

$$\begin{aligned} P\left(S_T \in I\right) = P&\left(\tau_1^{S,\gamma_1} \le T, \tau_1^{S,\gamma_1} < \tau_2^{S,\gamma_2}, S_T \in I\right) \\ &+ P\left(\tau_2^{S,\gamma_2} \le T, \tau_2^{S,\gamma_2} < \tau_1^{S,\gamma_1}, S_T \in I\right) \\ &+ P\left(\tau_1^{S,\gamma_1} > T, \tau_2^{S,\gamma_2} > T, S_T \in I\right), \end{aligned}$$

Given its similarity to the result for hitting an upper barrier before a lower barrier, we would like to recommend that this approach be considered for use in a future implementation of this method to price an actual deal.