Pricing Rainbow Partial Barrier Option

A rainbow partial barrier option is an option on two assets where one asset-the trigger asset-knocks in an European call or put on a second asset; such an option is therefore European. (Unless specified otherwise, all options are European style.) The adjective partial refers to the fact that the knock in or out period is shorter than the option tenor.

The adjective rainbow describes the fact that the option is on two assets and cannot be priced as equivalent option on a single asset. Rainbow partial barrier options come in eight “colors”: the trigger asset initially may be above or below the barrier knocking in or out a call or put on the other asset.

We implemented analytic model for pricing rainbow partial barrier options where a trigger asset knocks in or out an European call or put on a different asset.

Let $S1(t)$ denote the time t value of asset that may be called or put; let $S2(t)$ denote the time t value of the trigger asset. $S1(t)$ and $S2(t)$ satisfy the follow stochastic differential equations (sde’s) under risk neutral measure:

$$dS_1(t) = \mu_1(t)S_1(t)dt + \sigma_1(t)S_1(t)dW_1(t)\ , \qquad (1)$$

$$dS_2(t) = \mu_2 S_2(t)dt + \sigma_2 S_2(t)dW_2(t)\ , \qquad (2)$$

$$\mathrm{cov}\big(W_1(T), W_2(t)\big) = \rho \min(T, t)\ , \qquad (3)$$

Where

·$W t 1$ ( ) and $W t 2$ ( ) are standard Brownian motions with correlation r (-1 £ r £ 1);

· m1 ( ) t is the instantaneous drift for asset, and s1 (t) is the instantaneous volatility on asset one; these quantities are deterministic functions of time;

· m2 and s 2 for the trigger asset are interpreted similarly according to the corresponding quantities for the first asset.

Note that m1 ( ) t can be written as m1 1 1 (t) = r (t) - q (t) where $r t 1$ ( ) is the instantaneous risk-free rate $q t 1$ ( ) is the instantaneous dividend rate for asset and similarly for m2 for the trigger asset.

Let t denote the barrier tenor and let T denote the option tenor where 0 £ t £ T . The value of a rainbow up-and-out partial barrier put option (UOP) at option expiry is given by

$$X_{UOP}(T) = \left(1 - I_{[0,H]}\left(M_t\right)\right) \cdot \max(K - S_1(T), 0) \qquad (4)$$

Where

· Mt is the maximum of $S u 2$ ( ) over [0, t];

· K is strike level and $S T 1$ ( ) is the level of asset one at option expiry;

· H is the barrier level, and

· the initial value, S2 (0) , of the trigger asset satisfies $S H 2$ (0) < .

The value of a rainbow down-and-in partial barrier call option (DIC) at option expiry is given by

$$X_{DIC}(T) = I_{[0,H]}(m_t) \cdot \max(S_1(T) - K, 0) \qquad (5)$$

where *mt* is the maximum of *S u* 2 ( ) over [0, *t*] and

· *mt* is the minimum of *S u* 2 ( ) over [0, *t*], and

· *S*2 (0) satisfies *S H* 2 (0) > .

The other variables in equation (5) have the same interpretation as those in equation (4). The function [ ] *I* (*x*) 0,*H* appearing in equations (4) and (5) is defined as follows:

$$I_{[0,H]}(x) \equiv \begin{cases} 1, & 0 \le x \le H \\ 0, & otherwise \end{cases}$$

The current value of the option (see https://finpricing.com/lib/FxCompound.html) is obtained using the martingale representation for options. For example, the current value of a rainbow up-and-out partial barrier put option is given by taking the risk neutral expectation of *XUOP* using the money market account as numeraire:

$$V_{UOP}^{0} = \exp\left(-\bar{r}_1(T) \cdot T\right) E^{0}\left[X_{UOP}\right] \qquad (6)$$

Where

$$\bar{r}_1(T) = \frac{1}{T}\int_0^T r_1(s)\,ds \, .$$

The other option flavors are obtained by exchanging put payoffs for call payoffs and vice versa; the other barrier flavors are obtained from various parity relations. A plain vanilla put (call) can be replicated as long position in a rainbow partial barrier up-and-out put (call) and up-and-in put (call). We provide analytic results for all colors of the rainbow.

S t 1 ( ) and *S t* 2 ( ) satisfy the follow stochastic differential equations (sde's) under risk neutral measure

$$
\begin{aligned}
dS_1(t) &= \mu_1(t)S_1(t)dt + \sigma_1(t)S_1(t)dW_1(t) \\
dS_2(t) &= \mu_2 S_2(t)dt + \sigma_2 S_2(t)dW_2(t)
\end{aligned}
$$

Let *t* denote the barrier tenor and let *T* denote the option tenor where 0 £ *t* £ *T* . Let *K* denote the strike level, *H* the barrier level, and *S t* 1 ( ) *S t* 2 ( ) . We then obtained the following analytic result for rainbow partial barrier options:

$$
\begin{aligned}
V(\eta,\varphi,\zeta) = {} & \zeta\eta S_1(0)\exp\left(-\bar{q}_1(T)T\right)\Big\{ M\Big[\eta d_1, \varphi e_1; -\eta\varphi\rho\sqrt{t\,v^2(t)/T\,\bar{v}^2(T)}\Big] - \\
& \exp\left[\frac{2\left(r_2 - q_2 - \sigma_2^2/2 + \rho\sigma^2(t)\right)\ln\left(H/S_1(0)\right)}{\sigma_2^2}\right] M\Big[\eta d_1, \varphi e_1; -\eta\varphi\rho\sqrt{t\,v^2(t)/T\,\bar{v}^2(T)}\Big]\Big\} - \\
& \zeta\eta K\exp\left(-\bar{r}_1(T)T\right)\Big\{ M\Big[\eta d_2, \varphi e_2; -\eta\varphi\rho\sqrt{t\,v^2(t)/T\,\bar{v}^2(T)}\Big] - \\
& \exp\left[\frac{2\left(r_2 - q_2 - \sigma_2^2/2\right)\ln\left(H/S_1(0)\right)}{\sigma_2^2}\right] M\Big[\eta d_1, \varphi e_1; -\eta\varphi\rho\sqrt{t\,v^2(t)/T\,\bar{v}^2(T)}\Big]\Big\} - \\
& \frac{(\xi-1)}{2}\left\{\eta S_1(0)\exp\left(-\bar{q}_1(T)T\right)N\left[\eta d_1\right] - \eta K\exp\left(-\bar{r}_1(T)T\right)N\left[\eta d_2\right]\right\}
\end{aligned}
$$