Pricing Compound Option

A vanilla compound option is defined as a European vanilla option upon another European vanilla option, which may be called underlying vanilla option. There are four types of compound options: call-on-call, call-on-put, put-on-call and put-on-put. Due to the call-put parity, basically, we only need to consider call-on-call and call-on-put. In this report, under the assumption that the asset price, which is the underlyer of the underlying option, follows geometrical Brownian motion and that risk-free short rate, dividend yield and volatility are deterministic, we present Black-Scholes/Merton's analytical close form pricing formula for vanilla compound options.

For a European compound call option, the payoff at maturity is given by the maximum of the basket level less the strike, and zero. The option price, given by the discounted expected value of the payoff above, is calculated from a Gauss-Hermite quadrature. The model also provides various hedge ratios, which are approximated using finite differencing but based on parallel shifts to the respective independent variables.

A compound basket option specification includes :

- the option flavour, European call or put,
- the option maturity, T ,
- the option strike, X ,
- a basket of futures options where the ith such option is specified by
- the futures option flavour, European call or put,
- the futures option maturity, T_i ,
- the futures option strike, X_i , and
- a corresponding basket weight, w_i .

The i th underlying price process, f , follows driftless

Let $\{S_t\}$ be the price of a given asset which follows the following SDE

$$\mathrm{d}S_t = \mu_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t \ , \qquad t > 0 \ ,$$

where *{Wt}* is a standard R-valued Wiener process, *¹t* and *¾t* are deterministic drift term and volatility, respectively. Let *rt* be the deterministic risk-free short interest rate. For *t* < *s*, let us define

$$\tilde{\mu}(t,s) = \int_t^s \mu \mathrm{d}\tau \ , \quad \tilde{\sigma}(t,s) = \sqrt{\int_t^s \sigma^2 \mathrm{d}\tau} \ , \quad \tilde{r}(t,s) = \int_t^s r \mathrm{d}\tau \ .$$

The discounting factor from *s* back to *t*, denoted by df(*t; s*), can be written as

$$\mathrm{df}(t,s) = \exp[-\tilde{r}(t,s)] \ ,$$

and a forward price (see https://finpricing.com/FinPricing-ProductBrochure.pdf) seen at *t* matured at *s*, denoted by *F*(*t; s*), can be written as

$$F(t,s) = S_t \cdot \exp(\tilde{\mu}(t,s)) \ .$$

Let *t* < *T* and *ST* = *eZT* . Then we have

$$Z_T \ \sim_t \ \mathrm{N}\left(m_T = \ln S_t + \tilde{\mu}(t,T) - \frac{\tilde{\sigma}^2(t,T)}{2} \ , \ v_T = \tilde{\sigma}^2(t,T) \right) \ , \qquad t \le T \ .$$

Further, Let *t* < *T*1 < *T*2, *ST*1 = *eZ*1 and *ST*2 = *eZ*2 . Then, *Z*1 *»t* N(*m*1*; v*1), *Z*2 *»t* N(*m*2*; v*2), and relative to the time of *t*, (*Z*1*;Z*2) is jointly normal distributed with the following correlation coefficient

$$\rho = \mathrm{Corr}_t(Z_1, Z_2) = \frac{\tilde{\sigma}(t,T_1)}{\tilde{\sigma}(t,T_2)} \ .$$

Let $f(¢; ¢)$ be the joint density function of $(Z1;Z2)$ relative to time t, $f1(¢)$ is the density function of $Z1$ relative to time t and $f2j1(¢ ; ¢)$ is the density function of $Z2$ conditional on $Z1$ relative to time t. Clearly, we have

$$f(z_1, z_2) = f_1(z_1) \cdot f_{2|1}(z_2; z_1) \ ,$$

where

$$\begin{aligned} f(z_1) &= \frac{1}{\sqrt{2\pi v_1}} \cdot \exp\left[-\frac{1}{2}\frac{(z_1 - m_1)^2}{v_1}\right] \ , \\ f(z_1, z_2) &= \frac{1}{2\pi\sqrt{v_1 v_2 (1-\rho^2)}} \cdot \\ &\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(z_1-m_1)^2}{v_1} - 2\rho\frac{(z_1-m_1)(z_2-m_2)}{\sqrt{v_1 v_2}} + \frac{(z_2-m_2)^2}{v_2}\right]\right\} \ . \end{aligned}$$

Let T be a maturity of the compound option with a strike $K > 0$, $T1 > T$ be the maturity of the underlying option with a strike $K1 > 0$ and a call-put index °1. Then the compound option payoff at the maturity of T becomes

$$[p(T, S_T; K_1, T_1, \gamma_1) - K]^+ \ .$$

We have

$$p_c(t, S_t; K, \Gamma) = \mathrm{df}(t, T) \cdot \mathrm{E}_t\left[\{p(T, S_T; K_1, T_1, \gamma_1) - K\}^+\right] \ .$$

After substituting

$$p(T, S_T; K_1, T_1, \gamma_1) = \mathrm{df}(T, T_1) \cdot \mathrm{E}_T\left[\gamma_1 \cdot (S_{T_1} - K_1)^+\right]$$

In the following section, we will obtain analytical close form pricing formulae for the call-on-call and call-on-put compound options.

The price dynamic follows:

$$\frac{\partial}{\partial S} p(T, S; K_1, T_1, 1) = \mathrm{df}(T, T_1) \exp[\tilde{\mu}(T, T_1)] \Phi_1(D) > 0 \ ,$$

$$D = \frac{\ln \dfrac{F(T, T_1)}{K_1} + \dfrac{\tilde{\sigma}^2(T, T_1)}{2}}{\tilde{\sigma}(T, T_1)} \ .$$