# Calculating Average Volatility

## 1. Introduction

This paper presents a model for compute average volatility and correlation. Since an arithmetic average of log-normally distributed variables is not log-normal, We approximate the arithmetic average by matching its first and second moments with those of a log-normal variable. *We generate* the volatility of a log-normal variable that approximates an arithmetic average of asset prices. We also calculate the correlation between two log-normal variables chosen to match that between two arithmetic averages of asset prices.

## 2. Definition

We consider an arithmetic average of futures prices, specified with

- a set of $N$ observation points, $\{t_i | i = 1, \ldots, N\}$, where $0 < t_1 < \ldots < t_N = T$,
- a price at time $t_i$ of a futures contract maturing at time T, $F_T(t_i)$,
- a set of $N$ asset weights, $\alpha_i$ , where $1 \le i \le N$ .

The arithmetic average of futures prices is defined by

$$\mathrm{Avg}_1 = \sum_{\mathrm{i=1}}^{\mathrm{N}} \alpha_i \ F_T(t_i) \quad \textbf{\textit{(1)}}$$

The random variable $\text{Avg}_1$ is not log-normally distributed. However, we approximate the variable $\text{Avg}_1$ with a log-normal variable. Here, the implementation returns the volatility of the approximating log-normal random variable.

Given the same set of $N$ observation points, $\{t_i | i = 1,\ldots,N\}$, where $0 < t_1 <\ldots< t_N = T$, we also consider another arithmetic average of futures prices, $\text{Avg}_2$, specified with

- price at time $t_i$ of a futures contract maturing at time T, $V_T(t_i)$,
- set of $N$ asset weights, $\beta_i$ , where $1 \le i \le N$.

The function returns the constant instantaneous correlation coefficient between the Brownian motions driving the respective futures prices, to match the correlation between $\text{Avg}_1$ and $\text{Avg}_2$.

# 3 Valuation Methodology

## 3.1 The Underlying Process:

We assume that each futures price process $\{F_T(t) | t \in [0,T]\}$ satisfies a risk-neutral stochastic differential equation of the form

$$\frac{dF_T(t)}{F_T(t)} = \sigma \; dW_t \;, \textbf{(2)}$$

where

- $\sigma$ is a constant volatility parameter,
- $W$ is a standard Brownian Motion.

We similarly assume that

$$\frac{dV_T(t)}{V_T(t)} = \varpi \, dB_t \, ,$$

where

- $\varpi$ is a constant volatility parameter,
- $B$ is a standard Brownian Motion.

Here we assume that $B$ and $W$ have a constant, instantaneous correlation coefficient $\rho$.

## 3.2 Average Volatility:

We consider the random variable, $\mathrm{Avg}_1 = \sum_{\mathrm{i=1}}^{\mathrm{N}} \alpha_i \; F_T(t_i)$, where $0 \le t_1 < \ldots < t_N < T$.

$\mathrm{Avg}_1$ is not log-normally distributed. However, we approximate $\mathrm{Avg}_1$ with a log-normal variable, $X_T$, where the process $X$ satisfies

$$\frac{dX(t)}{X(t)} = \mu_X \, dt + \sigma_X \, dW \, .$$

Our aim is to determine $\mu_X$ and $\sigma_X$ by matching the first two moments of $\mathrm{Avg}_1$ with those of $X_T$. This approach (see https://finpricing.com/FinPricing-ProductBrochure.pdf) is well known as the Levy approximation. In particular, we consider the system of nonlinear equations

Our aim is to compute $E[X(T)]$ and $E\left[X(T)^2\right]$, where $\left\{ \frac{dX(t)}{X(t)} = \mu_X \, dt + \sigma_X \, dW \right.$ ***.(e1)***

From ***(e1)***, we have $X(T) = X(0)\exp\left(\left(\mu_X - \frac{\sigma_X^{\ 2}}{2}\right)T + \sigma_X\left(W_T - W_{t0}\right)\right)$,

$$X(T)^2 = X(0)^2 \exp\left(2\left(\mu_X - \frac{\sigma_X^{\ 2}}{2}\right)T + 2\sigma_X\left(W_T - W_{t_0}\right)\right).$$

Using the Laplace Transform property for Gaussian Variables, we get

$$E\left[\exp\left(\sigma_X\left(W_T - W_{t0}\right)\right)\right] = \exp\left(\frac{\sigma_X^{\ 2}}{2}(T - t_0)\right).$$

Finally, we have:

- $\boxed{E[X(T)] = X(t_0)\exp\left((\mu_X)(T - t_0)\right)}$,
- $\boxed{E\left[X(T)^2\right] = X(t_0)^2 \exp\left(\left(2\mu_X + \sigma_X^{\ 2}\right)(T - t_0)\right)}$.

Choosing $X_0 = E[\mathrm{Avg}_1]$, the solution of the above system is

$$\begin{cases} \mu_X = 0 \\ \sigma_X^{\ 2} T = \ln\left(\dfrac{E\left[\mathrm{Avg}_1^{\ 2}\right]}{\left(\mathrm{E}[\mathrm{Avg}_1]\right)^2}\right) \end{cases}, \textbf{\textit{(4)}}$$

where

- the computation of the first two moments of $\mathrm{Avg}_1$, $E[\mathrm{Avg}_1]$ and $E\left[\mathrm{Avg}_1^{\ 2}\right]$, is completely performed below,
- $\sigma_X$ is the average, or Levy, volatility, which is the value returned by the function.

Let $F_T(t_i)$ be the price at time $t_i$ of a futures contract maturing at T.

We consider the random variable, $\mathrm{Avg} = \sum_{\mathrm{i=1}}^{\mathrm{N}} \alpha_i \; F_{T_i}(t_i)$,

where

$F_{T_i}(t_i)$ is the price at time $t_i$ of a futures contract maturing at $T_{t_i}$,

$0 < t_1 < \ldots < t_N$,

$t_i < T_i$, $1 \le i \le N$,

and

$$F_{T_i}(t_i) = F_{T_i}(t_0)\exp\left(\frac{-\left(\sigma_{t_i}\right)^2}{2}\left(t_i - t_0\right) + \sigma_{t_i}\left(W_{t_i} - W_{t_0}\right)\right).$$

Our aim is to compute $E[\mathrm{Avg}]$ and $E[\mathrm{Avg}^2]$.

- Using the Laplace Transform property for Gaussian random variables, we directly have:

$$\boxed{E[\mathrm{Avg}] = \sum_{i=1}^{N} \alpha_i \; F_{T_i}(t_0)}.$$

- On the other hand, we can also express $\mathrm{Avg}^2$ as

$$\mathrm{Avg}^2 = \sum_{j=1}^{N} \alpha_j \sum_{\mathrm{i=1}}^{\mathrm{N}} \alpha_i \; F_{T_i}(t_i) F_{T_j}(t_j).$$

Taking the expectation,

$$E\left[\mathrm{Avg}^2\right] = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \, \alpha_j \; E\left[F_{T_i}(t_i) F_{T_j}(t_j)\right],$$

where,

$$E\left[F_{T_i}(t_i)F_{T_j}(t_j)\right]=F_{T_i}(t_0)\,F_{T_j}(t_0)\exp\left(\frac{-\sigma_{t_i}^{\;2}}{2}(t_i-t_0)+\frac{-\sigma_{t_j}^{\;2}}{2}(t_j-t_0)\right)E\left[\exp\left(\sigma_{t_i}(W_{t_i}-W_{t_0})\right)\exp\left(\sigma_{t_j}(W_{t_j}-W_{t_0})\right)\right]$$

***(e)***

Now we set

$$\Delta=E\left[\,\exp\left(\sigma_{t_i}\left(W_{t_i}-W_{t_0}\right)\right)\exp\left(\sigma_{t_j}\left(W_{t_j}-W_{t_0}\right)\right)\right]\;.$$

We have

$$\Delta=E\left[\;E\left[\exp\left(\sigma_{t_i}\left(W_{t_i}-W_{t_0}\right)\right)\exp\left(\sigma_{t_j}\left(W_{t_j}-W_{t_0}\right)\right)\;\middle|\;\Im_{t_i}\right]\right],$$

where $\Im_{t_i}$ is the filtration generated by $\{W_u\,,0\le u\le t_i\}$.

Because $W_{t_i}-W_{t_0}$ is $\Im_{t_i}$-measurable, we have

$$\Delta=E\left[\,\exp\left(\sigma_{t_i}\left(W_{t_i}-W_{t_0}\right)\right)E\left[\exp\left(\sigma_{t_j}\left(W_{t_j}-W_{t_i}\right)\right)\exp\left(\sigma_{t_j}\left(W_{t_i}-W_{t_0}\right)\right)\;\middle|\;\Im_{t_i}\right]\right],$$

then,

$$\Delta=E\left[\,\exp\left(\left(\sigma_{t_i}+\sigma_{t_j}\right)\left(W_{t_i}-W_{t_0}\right)\right)\;E\left[\,\exp\left(\sigma_{t_j}\left(W_{t_j}-W_{t_i}\right)\right)\;\middle|\;\Im_{t_i}\right]\right].$$

Now, we use the fact that Brownian Motion has independent increments. Therefore, $W_{t_j}-W_{t_i}$ is independent of $\Im_{t_i}$. This leads us to express $\Delta$ as:

$$\Delta=E\left[\,\exp\left(\left(\sigma_{t_i}+\sigma_{t_j}\right)\left(W_{t_i}-W_{t_0}\right)\right)\;E\left[\,\exp\left(\sigma_{t_j}\left(W_{t_j}-W_{t_i}\right)\right)\right]\right].$$

We note that the term $E\left[\exp\left(\sigma_{t_j}\left(W_{t_j}-W_{t_i}\right)\right)\right]$ is not a random variable, it can be taken out of the bigger expectation.

Therefore, we have

$$\Delta = E\left[\exp\left(\left(\sigma_{t_i}+\sigma_{t_j}\right)\left(W_{t_i}-W_{t_0}\right)\right)\right] E\left[\sigma_{t_j}\left(W_{t_j}-W_{t_i}\right)\right].$$

Using again the Laplace Transform property for Gaussian Variables, we finally get:

$$\Delta = \exp\left(\frac{\left(\sigma_{t_i}+\sigma_{t_j}\right)^2}{2}\left(t_i-t_0\right)\right)\exp\left(\frac{\sigma_{t_j}^{\ 2}}{2}\left(t_j-t_i\right)\right).$$

We substitute $\Delta$ into equation (e) above:

$$E\left[S_{t_i}\,S_{t_j}\right]=F_{T_i}\left(t_0\right)F_{T_j}\left(t_0\right)\exp\left(\sigma_{t_i}\,\sigma_{t_j}\left(t_i-t_0\right)\right), \text{ if } t_i \le t_j.$$

We can finally conclude:

$$\boxed{E\left[\mathrm{Avg}^2\right]=\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\,\alpha_j\,E\left[F_{T_i}\left(t_i\right)F_{T_j}\left(t_j\right)\right]=\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\,\alpha_j\,F_{T_i}\left(t_0\right)F_{T_j}\left(t_0\right)\exp\left(\sigma_{t_i}\sigma_{t_j}\left(\mathrm{Min}\left(t_i,t_j\right)-t_0\right)\right)}$$