Generalized fluid models of the Braginskii-type

(Why does the Braginskii viscosity and heat flux “explode”, when transitioning from the solar chromosphere to corona?)

Peter Hunana[1]

T. Passot[2], E. Khomenko[1], D. Martínez-Gómez[1], M. Collados[1], A. Tenerani[3], G. P. Zank[4], Y. Maneva[5], M. L. Goldstein[6], G. M. Webb[4]

(1) Instituto de Astrofisica de Canarias (IAC), Universidad de La Laguna, Tenerife, Spain

(2) Université Côte d’Azur, Observatoire de la Côte d’Azur, CNRS, Nice, France

(3) Department of Physics, University of Texas, Austin, TX, USA

(4) Center for Space Plasma and Aeronomic Research, University of Alabama, Huntsville, AL, USA

(5) Royal Observatory of Belgium, Brussels, Belgium

(6) Space Science Institute, Boulder, CO, USA

ESPOS 2022

Fluid models are constructed by integration of the Boltzmann equation over the velocity space (we use the cgs units):

$$\frac{\partial f_a}{\partial t} + \boldsymbol{v} \cdot \nabla f_a + \left[\boldsymbol{G} + \frac{eZ_a}{m_a}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right)\right] \cdot \nabla_v f_a = C(f_a)$$

distribution function f(x,v)

kinetic velocity

gravitational acceleration

electric field

magnetic field

collisional operator (for now unspecified)

"a" = the species index, Z_a = charge, c = the speed of light

One can define:

number density $n_a = \int f_a d^3v$

fluid velocity $\boldsymbol{u}_a = (1/n_a) \int \boldsymbol{v} f_a d^3v$

fluctuating (peculiar) velocity $\boldsymbol{c}_a = \boldsymbol{v} - \boldsymbol{u}_a$

pressure tensor $\bar{\bar{\boldsymbol{p}}}_a = m_a \int \boldsymbol{c}_a \boldsymbol{c}_a f_a d^3v$ $\qquad \bar{\bar{\boldsymbol{p}}}_a = p_a \bar{\bar{\boldsymbol{I}}} + \bar{\bar{\boldsymbol{\Pi}}}_a^{(2)}$

stress-tensor (= viscosity-tensor, it is a 2[nd] order fluid moment and we use superscript (2) to differentiate it from higher-order stress-tensors)

scalar pressure $p_a = \frac{1}{3}\mathrm{Tr}\bar{\bar{\boldsymbol{p}}}_a = \frac{m_a}{3}\int |\boldsymbol{c}_a|^2 f_a d^3v$ $\qquad \bar{\bar{\boldsymbol{\Pi}}}_a^{(2)} = m_a \int \left(\boldsymbol{c}_a \boldsymbol{c}_a - \frac{\bar{\bar{\boldsymbol{I}}}}{3}|\boldsymbol{c}_a|^2\right) f_a d^3v$

heat flux vector $\vec{\boldsymbol{q}}_a = \frac{m_a}{2}\int \boldsymbol{c}_a |\boldsymbol{c}_a|^2 f_a d^3v$

factor of 1/2 by purely historical convention

---

momentum exchange rates $\boldsymbol{R}_a = m_a \int \boldsymbol{v} C(f_a) d^3v$

energy exchange rates $Q_a = \frac{m_a}{2}\int |\boldsymbol{c}_a|^2 C(f_a) d^3v$

We do not consider ionization and recombination and the collisional operator conserves the number of particles

$$\int C(f_a) d^3v = 0$$

Direct integration of the Boltzmann equation then yields the well-known evolution equations

$$\frac{d_a}{dt}n_a + n_a \nabla \cdot \boldsymbol{u}_a = 0;$$

density equation

$$\frac{d_a}{dt}\boldsymbol{u}_a + \frac{1}{\rho_a}\nabla \cdot \bar{\bar{\boldsymbol{p}}}_a - \boldsymbol{G} - \frac{eZ_a}{m_a}\Big(\boldsymbol{E} + \frac{1}{c}\boldsymbol{u}_a \times \boldsymbol{B}\Big) = \frac{\boldsymbol{R}_a}{\rho_a};$$

momentum equation

$$\frac{d_a}{dt}p_a + \frac{5}{3}p_a \nabla \cdot \boldsymbol{u}_a + \frac{2}{3}\nabla \cdot \vec{\boldsymbol{q}}_a + \frac{2}{3}\bar{\bar{\boldsymbol{\Pi}}}^{(2)}_a : (\nabla \boldsymbol{u}_a) = \frac{2}{3}Q_a,$$

(scalar) pressure equation

we use convective derivative $d_a/dt = \partial/\partial t + \boldsymbol{u}_a \cdot \nabla$

These equations are very easy to derive. What is highly non-trivial, is to obtain expressions for the quantities Pi, q, R, Q. These are given by the famous paper of **Braginskii (1965)**, however, there are several problems:

1) Results are given only for a plasma consisting of one ion species and electrons, so called **one ion-electron plasma**. (The paper of Braginskii also contains Section 7 about multi-component plasmas that is often implicitly cited in the solar literature, but this section should be viewed as heuristic from a perspective that no heat fluxes or stress tensors were calculated).

2) Results are given in a quasi-static approximation (highly-collisional regime), and “explode”/become divergent in a weakly-collisional regime (e.g. solar corona) as the collisional frequency decreases (nu is in the denominator, Braginskii uses tau=1/nu). Examples are:

parallel ion viscosity: $\eta_0^a = 0.960\dfrac{p_a}{\nu_{aa}}$

parallel electron heat conductivity (thermal heat flux): $\kappa_\parallel^e = 3.1616\dfrac{p_e}{m_e\nu_{ei}}$

How is the Braginskii model derived ?

There are two major classical routes (with sub-variations):

1) **Chapman-Enskog expansions** (Chapman & Cowling 1939), originally developed for neutral gases, used by Braginskii to calculate magnetized plasma, spherical coordinates are used and the distribution function is expanded in Laguerre-Sonine polynomials. Typically expressions in a quasi-static approximation are obtained directly (q=something, Pi=something).

2) **moment method of Grad** (1949, 1958), first used by Balescu (1988) to recover Braginskii, usual Cartesian (rectangular) coordinates are used and the distribution function is expanded in Hermite polynomials. Evolution equations (dq/dt, dPi/dt) are obtained first, and the quasi-static approximation (i.e. the highly-collisional limit) is applied only later.

(There also exists a third more modern route with the **projection operator**, Krommes 2018)

As pointed out for example by Balescu, both polynomials are directly related and both methods must yield the same results:

Hermite polynomials

Laguerre-Sonine polynomials (scalars)

Vectors:
(yields heat fluxes)

$$H_i^{(2n+1)}(\tilde{\boldsymbol{c}}) = \sqrt{\frac{3}{2}}\tilde{c}_i L_n^{(3/2)}(\frac{\tilde{c}^2}{2});$$

Matrices:
(yields viscosities)

$$H_{ij}^{(2n)}(\tilde{\boldsymbol{c}}) = \sqrt{\frac{15}{8}}(\tilde{c}_i\tilde{c}_j - \frac{\tilde{c}^2}{3}\delta_{ij})L_{n-1}^{(5/2)}(\frac{\tilde{c}^2}{2})$$

We use tilde on c as a shortcut for the normalized fluctuating velocity

$$\tilde{\boldsymbol{c}} = \sqrt{m_a/T_a}\boldsymbol{c}_a$$

The moment method of Grad
Expansion in Hermite polynomials

- considering only viscosity-tensors and heat fluxes, and neglecting fully contracted scalar perturbations and higher-order tensorial "anisotropies" (as Balescu 1988 calls them):

Maxwellian

Viscosity-tensors (matrices), traceless, 5 independent components

Heat flux vectors, 3 independent comp.

$$f_a = f_a^{(0)}(1+\chi_a); \qquad \chi_a = \sum_{n=1}^{N}\left[h_{ij}^{(2n)}H_{ij}^{(2n)} + h_i^{(2n+1)}H_i^{(2n+1)}\right]$$

perturbation around a Maxwellian

h = Hermite moments

H = Hermite polynomials

N=0; $\chi_a = 0$; **5-moment model**, with evolution equations for density, fluid velocity & scalar pressure (viscosity-tensors and heat fluxes are zero).

N=1; **13-moment model**, with evolution equations for the usual viscosity-tensor and heat flux vector. However, in some cases the model is not sufficiently precise (see the next slide). Multi-fluid models of **Burgers 1969 - Schunk 1977.**

N=2; **21-moment model**, with evolution equations for two viscosity-tensors and two heat flux vectors. In a quasi-static approximation (no d/dt), it can be shown that the model is equivalent to **Braginskii**.

General N; corresponds to (5+8N)-moment model

N=∞; **Spitzer-Harm 1953**, the heat flux perturbation was found numerically, the work is criticized by Balescu (no magnetic field present, no viscosity, does not satisfy the Onsager symmetry - frictional heat flux incorrect).

Of course, the model of Braginskii can be generalized in many different ways.
Naturally, one might focus at the case of a one ion-electron plasma considered by Braginskii, and increase the order of N to study convergence of transport coefficients with higher-order Laguerre (Hermite) schemes. Several studies of this kind were done in the past.

Landshoff 1949, 1951 (before Braginskii)- calculated several transport coefficients from N=1 to N=4.
Kaneko 1960 - improved numerical accuracy of Landshoff (more digits) and done up to N=5.
Kaneko & Taguchi 1978, Kaneko Yamayo 1980 - up to N=49.
Ji & Held 2013 - studied convergence of all the transport coefficients with up to N=160.

Example: parallel electron thermal heat flux $\vec{\boldsymbol{q}}_e^T = -\kappa_\parallel^e \nabla_\parallel T_e$
with electron thermal conductivity:

$$\kappa_\parallel^e = \gamma_0 p_e/(m_e \nu_{ei})$$

Burgers 1969 - Schunk 1977 becomes very imprecise, especially for large Z_i

$\parallel$ heat conductivity $\kappa_\parallel^e$	$Z_i = 1$	$Z_i = 2$	$Z_i = 3$	$Z_i = 4$	$Z_i = 16$	$Z_i = \infty$
Burgers-Schunk (N=1)	1.34	1.58	1.68	1.73	1.87	1.92
Killie et al.	3.92	5.25	5.92	6.33	7.47	7.95
Braginskii (N=2)	3.1616	4.890	6.064	6.920	10.334	12.471
Landshoff (N=4)	3.178	4.902	6.069			13.572
Spitzer-Härm (N=∞)	3.203	4.960		6.983	10.629	13.581

Final value from Kenako et al. 1980 and Ji & Held 2013 is: 3.2031
and Spitzer-Harm 1953 is correct.

Here we focus at the case N=2 (i.e. equivalent to Braginskii), with the goal to extract more physical information from that scheme.

Several various multi-fluid descriptions for the case N=2 of Braginskii have been considered in the past:

Hinton 1983
Zhdanov 2002 (orig. publ. 1982)
Ji & Held 2006 (who actually consider general N)
Symakov & Molvig 2014, 2016
Hirshman & Sigmar 1977, 1981 - toroidal geometry applicable to tokamaks

Our multi-fluid model seems to be very close to **Zhdanov 2002**, Chapter 8.1, but we did not verify full equivalence because of his puzzling notation. Even if equivalence is eventually shown for the case of small temperature difference between ions, we consider a more general case where temperatures of all the species are arbitrary. Our clear formulation with fluid moments (instead of Hermite moments) might be also easier to implement into existing numerical codes.

Arbitrary temperatures (and masses) were also considered by **Ji and Held 2006**, but we did not verify equivalence with their model either...

Multi-fluid generalization of Braginskii (21-moment model)

Let us first present formulation valid for any collisional operator C(f_a).
Our model is formulated with two heat flux vectors $\vec{q}_a;\ \vec{X}_a^{(5)}$ and two viscosity-tensors $\bar{\bar{\Pi}}_a^{(2)};\bar{\bar{\Pi}}_a^{(4)}$

$$\vec{X}_a^{(3)} = m_a \int \boldsymbol{c}_a |\boldsymbol{c}_a|^2 f_a d^3v = 2\vec{q}_a; \qquad \vec{X}_a^{(5)} = m_a \int \boldsymbol{c}_a |\boldsymbol{c}_a|^4 f_a d^3v,$$

usual heat flux vector

“heat flux” vector of the 5th-order fluid moment (free wording)

$$\bar{\bar{\Pi}}_a^{(2)} = m_a \int \Big(\boldsymbol{c}_a\boldsymbol{c}_a - \frac{\bar{\bar{I}}}{3}|\boldsymbol{c}_a|^2\Big) f_a d^3v; \qquad \bar{\bar{\Pi}}_a^{(4)} = m_a \int \Big(\boldsymbol{c}_a\boldsymbol{c}_a - \frac{\bar{\bar{I}}}{3}|\boldsymbol{c}_a|^2\Big)|\boldsymbol{c}_a|^2 f_a d^3v,$$

usual viscosity-tensor

“viscosity-tensor” of the 4th-order fluid moment (free wording)

Each of these moments has its own evolution equation. To describe right-hand-sides of these equations, we define (tensorial) collisional contribution

usual momentum exchange rates

$$\boldsymbol{R}_a = m_a \int \boldsymbol{v} C(f_a) d^3v; \qquad Q_a = \frac{m_a}{2} \int |\boldsymbol{c}_a|^2 C(f_a) d^3v;$$

usual energy exchange rates

$$\bar{\bar{\boldsymbol{Q}}}_a^{(2)} = m_a \int \boldsymbol{c}_a\boldsymbol{c}_a C(f_a) d^3v; \qquad \bar{\bar{\boldsymbol{Q}}}_a^{(3)} = m_a \int \boldsymbol{c}_a\boldsymbol{c}_a\boldsymbol{c}_a C(f_a) d^3v;$$

$$\bar{\bar{\boldsymbol{Q}}}_a^{(4)} = m_a \int \boldsymbol{c}_a\boldsymbol{c}_a\boldsymbol{c}_a\boldsymbol{c}_a C(f_a) d^3v; \qquad \bar{\bar{\boldsymbol{Q}}}_a^{(5)} = m_a \int \boldsymbol{c}_a\boldsymbol{c}_a\boldsymbol{c}_a\boldsymbol{c}_a\boldsymbol{c}_a C(f_a) d^3v,$$

For each specie, the entire model is then given by "basic" evolution equations (shown before):

$$\frac{d_a}{dt} n_a + n_a \nabla\cdot\boldsymbol{u}_a = 0;$$

$$\frac{d_a}{dt}\boldsymbol{u}_a + \frac{1}{\rho_a}\nabla\cdot\bar{\bar{\boldsymbol{p}}}_a - \boldsymbol{G} - \frac{eZ_a}{m_a}\Big(\boldsymbol{E} + \frac{1}{c}\boldsymbol{u}_a\times\boldsymbol{B}\Big) = \frac{\boldsymbol{R}_a}{\rho_a};$$

$$\frac{d_a}{dt}p_a + \frac{5}{3}p_a\nabla\cdot\boldsymbol{u}_a + \frac{2}{3}\nabla\cdot\vec{q}_a + \frac{2}{3}\bar{\bar{\Pi}}_a^{(2)} : (\nabla\boldsymbol{u}_a) = \frac{2}{3}Q_a,$$

which are accompanied by the evolution equations for the viscosity-tensors and heat flux vectors:

$$\frac{d_a\bar{\bar{\Pi}}^{(2)}_a}{dt}+\bar{\bar{\Pi}}^{(2)}_a\nabla\cdot\boldsymbol{u}_a+\Omega_a\big(\hat{\boldsymbol{b}}\times\bar{\bar{\Pi}}^{(2)}_a\big)^S+\big(\bar{\bar{\Pi}}^{(2)}_a\cdot\nabla\boldsymbol{u}_a\big)^S-\frac{2}{3}\bar{\bar{\boldsymbol{I}}}\big(\bar{\bar{\Pi}}^{(2)}_a:\nabla\boldsymbol{u}_a\big)$$
$$+\frac{2}{5}\Big[(\nabla\vec{\boldsymbol{q}}_a)^S-\frac{2}{3}\bar{\bar{\boldsymbol{I}}}\nabla\cdot\vec{\boldsymbol{q}}_a\Big]+p_a\bar{\bar{\boldsymbol{W}}}_a=\bar{\bar{\boldsymbol{Q}}}^{(2)\,\prime}_a\equiv\bar{\bar{\boldsymbol{Q}}}^{(2)}_a-\frac{\bar{\bar{\boldsymbol{I}}}}{3}\mathrm{Tr}\bar{\bar{\boldsymbol{Q}}}^{(2)}_a;$$

Evolution equation for the usual viscosity-tensor $\bar{\bar{\Pi}}^{(2)}_a$;

$$\frac{d_a\vec{\boldsymbol{q}}_a}{dt}+\frac{7}{5}\vec{\boldsymbol{q}}_a\nabla\cdot\boldsymbol{u}_a+\frac{7}{5}\vec{\boldsymbol{q}}_a\cdot\nabla\boldsymbol{u}_a+\frac{2}{5}(\nabla\boldsymbol{u}_a)\cdot\vec{\boldsymbol{q}}_a+\Omega_a\hat{\boldsymbol{b}}\times\vec{\boldsymbol{q}}_a+\frac{5}{2}p_a\nabla\Big(\frac{p_a}{\rho_a}\Big)$$
$$+\frac{1}{2}\nabla\cdot\bar{\bar{\Pi}}^{(4)}_a-\frac{5}{2}\frac{p_a}{\rho_a}\nabla\cdot\bar{\bar{\Pi}}^{(2)}_a-\frac{1}{\rho_a}(\nabla\cdot\bar{\bar{\boldsymbol{p}}}_a)\cdot\bar{\bar{\Pi}}^{(2)}_a$$
$$=\vec{\boldsymbol{Q}}^{(3)\,\prime}_a\equiv\frac{1}{2}\mathrm{Tr}\bar{\bar{\boldsymbol{Q}}}^{(3)}_a-\frac{5}{2}\frac{p_a}{\rho_a}\boldsymbol{R}_a-\frac{1}{\rho_a}\boldsymbol{R}_a\cdot\bar{\bar{\Pi}}^{(2)}_a;$$

Evolution equation for the usual heat flux vector $\vec{\boldsymbol{q}}_a$;

$$\frac{d_a}{dt}\bar{\bar{\Pi}}^{(4)}_a+\frac{1}{5}\Big[(\nabla\vec{\boldsymbol{X}}^{(5)}_a)^S-\frac{2}{3}\bar{\bar{\boldsymbol{I}}}(\nabla\cdot\vec{\boldsymbol{X}}^{(5)}_a)\Big]+\frac{9}{7}(\nabla\cdot\boldsymbol{u}_a)\bar{\bar{\Pi}}^{(4)}_a+\frac{9}{7}(\bar{\bar{\Pi}}^{(4)}_a\cdot\nabla\boldsymbol{u}_a)^S$$
$$+\frac{2}{7}\big((\nabla\boldsymbol{u}_a)\cdot\bar{\bar{\Pi}}^{(4)}_a\big)^S-\frac{22}{21}\bar{\bar{\boldsymbol{I}}}(\bar{\bar{\Pi}}^{(4)}_a:\nabla\boldsymbol{u}_a)+\Omega_a\big(\hat{\boldsymbol{b}}\times\bar{\bar{\Pi}}^{(4)}_a\big)^S+7\frac{p_a^2}{\rho_a}\bar{\bar{\boldsymbol{W}}}_a$$
$$-\frac{14}{5\rho_a}\Big[\big((\nabla\cdot\bar{\bar{\boldsymbol{p}}}_a)\vec{\boldsymbol{q}}_a\big)^S-\frac{2}{3}\bar{\bar{\boldsymbol{I}}}(\nabla\cdot\bar{\bar{\boldsymbol{p}}}_a)\cdot\vec{\boldsymbol{q}}_a\Big]$$
$$=\bar{\bar{\boldsymbol{Q}}}^{(4)\,\prime}_a\equiv\mathrm{Tr}\bar{\bar{\boldsymbol{Q}}}^{(4)}_a-\frac{\bar{\bar{\boldsymbol{I}}}}{3}\mathrm{Tr}\mathrm{Tr}\bar{\bar{\boldsymbol{Q}}}^{(4)}_a-\frac{14}{5\rho_a}\Big[(\boldsymbol{R}_a\vec{\boldsymbol{q}}_a)^S-\frac{2}{3}\bar{\bar{\boldsymbol{I}}}(\boldsymbol{R}_a\cdot\vec{\boldsymbol{q}}_a)\Big];$$

$$\frac{d_a}{dt}\vec{\boldsymbol{X}}^{(5)}_a+\nabla\cdot\bar{\bar{\Pi}}^{(6)}_a+\frac{9}{5}\vec{\boldsymbol{X}}^{(5)}_a(\nabla\cdot\boldsymbol{u}_a)+\frac{9}{5}\vec{\boldsymbol{X}}^{(5)}_a\cdot\nabla\boldsymbol{u}_a+\frac{4}{5}(\nabla\boldsymbol{u}_a)\cdot\vec{\boldsymbol{X}}^{(5)}_a+\Omega_a\hat{\boldsymbol{b}}\times\vec{\boldsymbol{X}}^{(5)}_a$$
$$+70\frac{p_a^2}{\rho_a}\nabla\Big(\frac{p_a}{\rho_a}\Big)-35\frac{p_a^2}{\rho_a^2}\nabla\cdot\bar{\bar{\Pi}}^{(2)}_a-\frac{4}{\rho_a}(\nabla\cdot\bar{\bar{\boldsymbol{p}}}_a)\cdot\bar{\bar{\Pi}}^{(4)}_a$$
$$=\vec{\boldsymbol{Q}}^{(5)\,\prime}_a\equiv\mathrm{Tr}\mathrm{Tr}\bar{\bar{\boldsymbol{Q}}}^{(5)}_a-35\frac{p_a^2}{\rho_a^2}\boldsymbol{R}_a-\frac{4}{\rho_a}\boldsymbol{R}_a\cdot\bar{\bar{\Pi}}^{(4)}_a.$$

Coupled with the evolution equations for the higher-order moments $\bar{\bar{\Pi}}^{(4)}_a$ $\vec{\boldsymbol{X}}^{(5)}_a$

The equations contain symmetric operator "S", which acts on a matrix as:

$$A^S_{ij}=A_{ij}+A_{ji}$$

rate-of-strain tensor

$$\bar{\bar{\boldsymbol{W}}}_a=(\nabla\boldsymbol{u}_a)^S-(2/3)\bar{\bar{\boldsymbol{I}}}\nabla\cdot\boldsymbol{u}_a;$$

cyclotron frequency: $\Omega_a=eZ_a|\boldsymbol{B}|/(m_ac)$

direction of magnetic field: $\hat{\boldsymbol{b}}=\boldsymbol{B}/|\boldsymbol{B}|$

The last equation is closed with a fluid closure (derived from a Hermite closure):

$$\bar{\bar{\Pi}}^{(6)}_a=m_a\int\Big(\boldsymbol{c}_a\boldsymbol{c}_a-\frac{\bar{\bar{\boldsymbol{I}}}}{3}|\boldsymbol{c}_a|^2\Big)|\boldsymbol{c}_a|^4f_ad^3v=18\frac{p_a}{\rho_a}\bar{\bar{\Pi}}^{(4)}_a-63\frac{p_a^2}{\rho_a^2}\bar{\bar{\Pi}}^{(2)}_a$$

The system represents a generalized model of Braginskii, where evolution equations for all the moments are fully non-linear and valid for a general collisional operator.

Collisional contributions calculated with the Landau collisional operator (i.e. Fokker-Planck operator, applicable to charges)

reduced mass $\longrightarrow$ $\mu_{ab} = \frac{m_a m_b}{m_a + m_b}$; $T_{ab} = \frac{m_a T_b + m_b T_a}{m_a + m_b}$, $\longleftarrow$ reduced temperature

- The collisional integrals are calculated in the semi-linear approximation, with an assumption that the differences in drift (fluid) velocities are much smaller than thermal velocities (the runaway effect is neglected). Wording semilinear means that expressions such as $(p_a/\rho_a)\vec{\boldsymbol{q}}_a$ are retained and not linearized with mean pressure/density values.

$$\epsilon = \frac{|\boldsymbol{u}_b - \boldsymbol{u}_a|}{\sqrt{v_{\mathrm{th}a}^2 + v_{\mathrm{th}b}^2}} \ll 1$$

- All the nonlinear quantities such as $\vec{\boldsymbol{q}}_a \cdot (\boldsymbol{u}_b - \boldsymbol{u}_a)$ or $|\boldsymbol{u}_b - \boldsymbol{u}_a|^2$ are thus neglected in the multi-fluid description. This is consistent with the multi-fluid models of Burgers (1969)-Schunk (1977).

Momentum exchange rates:

5-moment model

$$\boldsymbol{R}_a = \sum_{b\neq a} \nu_{ab} \Big\{ \rho_a(\boldsymbol{u}_b - \boldsymbol{u}_a) + \frac{\mu_{ab}}{T_{ab}}\Big[V_{ab(1)}\vec{\boldsymbol{q}}_a - V_{ab(2)}\frac{\rho_a}{\rho_b}\vec{\boldsymbol{q}}_b\Big] - \frac{3}{56}\Big(\frac{\mu_{ab}}{T_{ab}}\Big)^2\Big[\vec{\boldsymbol{X}}_a^{(5)} - \frac{\rho_a}{\rho_b}\vec{\boldsymbol{X}}_b^{(5)}\Big]\Big\},$$

21-moment model, both heat fluxes q and X^5 now enter and create the thermal force (thermal diffusion)

"mass-ratio coefficients":

$$V_{ab(1)} = \frac{(21/10)T_a m_b + (3/5)T_b m_a}{T_a m_b + T_b m_a}; \qquad V_{ab(2)} = \frac{(3/5)T_a m_b + (21/10)T_b m_a}{T_a m_b + T_b m_a}$$

---

collisional frequency $\nu_{ab} = \tau_{ab}^{-1} = \frac{16}{3}\sqrt{\pi}\frac{n_b e^4 Z_a^2 Z_b^2 \ln\Lambda}{m_a^2(v_{\mathrm{th}a}^2 + v_{\mathrm{th}b}^2)^{3/2}}\Big(1 + \frac{m_a}{m_b}\Big),$

Energy exchange rates: $Q_a = \sum_{b\neq a} 3\rho_a\nu_{ab}\frac{T_b - T_a}{m_a + m_b}$, with $|\boldsymbol{u}_b - \boldsymbol{u}_a|^2$ neglected as described above.

Viscosity-tensor exchange rates:

self-collisions

contains both viscosities (but no heat fluxes)

$$\bar{\bar{\boldsymbol{Q}}}_a^{(2)\,\prime} = -\frac{21}{10}\nu_{aa}\bar{\bar{\boldsymbol{\Pi}}}_a^{(2)} + \frac{9}{70}\nu_{aa}\frac{\rho_a}{p_a}\bar{\bar{\boldsymbol{\Pi}}}_a^{(4)} + \sum_{b\neq a}\frac{\rho_a\nu_{ab}}{m_a+m_b}\Big[-\hat{K}_{ab(1)}\frac{1}{n_a}\bar{\bar{\boldsymbol{\Pi}}}_a^{(2)} + \hat{K}_{ab(2)}\frac{1}{n_b}\bar{\bar{\boldsymbol{\Pi}}}_b^{(2)} + L_{ab(1)}\frac{\rho_a}{n_a p_a}\bar{\bar{\boldsymbol{\Pi}}}_a^{(4)} - L_{ab(2)}\frac{\rho_b}{n_b p_b}\bar{\bar{\boldsymbol{\Pi}}}_b^{(4)}\Big],$$

"mass-ratio coefficients"

$$\hat{K}_{ab(1)} = \frac{10T_a^2 m_a m_b^2 + 15T_a^2 m_b^3 + 35T_aT_b m_a^2 m_b + 42T_aT_b m_a m_b^2 + 10T_b^2 m_a^3 + 12T_b^2 m_a^2 m_b}{5(T_a m_b + T_b m_a)^2 m_a};$$

$$\hat{K}_{ab(2)} = \frac{6T_a^2 m_a m_b + 4T_a^2 m_b^2 + 21T_aT_b m_a^2 + 14T_aT_b m_a m_b - 5T_b^2 m_a^2}{5(T_a m_b + T_b m_a)^2};$$

$$L_{ab(1)} = \frac{3T_a m_b(2T_a m_a m_b + 3T_a m_b^2 + 7T_b m_a^2 + 8T_b m_a m_b)}{35(T_a m_b + T_b m_a)^2 m_a};$$

$$L_{ab(2)} = \frac{3m_a T_b(5T_a m_a + 4T_a m_b - T_b m_a)}{35(T_a m_b + T_b m_a)^2}.$$

Heat flux (3rd-order moment) exchange rates:

contains both heat fluxes (but no viscosities)

$$\vec{\boldsymbol{Q}}_a^{(3)\,\prime} = -\Big[2\nu_{aa} + \sum_{b\neq a}\nu_{ab}\hat{D}_{ab(1)}\Big]\vec{\boldsymbol{q}}_a + \sum_{b\neq a}\nu_{ab}\hat{D}_{ab(2)}\frac{\rho_a}{\rho_b}\vec{\boldsymbol{q}}_b + \Big[\frac{3}{70}\nu_{aa} + \sum_{b\neq a}\nu_{ab}\hat{E}_{ab(1)}\Big]\frac{\rho_a}{p_a}\vec{\boldsymbol{X}}_a^{(5)} - \sum_{b\neq a}\nu_{ab}\hat{E}_{ab(2)}\frac{\rho_b}{p_b}\frac{\rho_a}{\rho_b}\vec{\boldsymbol{X}}_b^{(5)} - p_a\sum_{b\neq a}\nu_{ab}(\boldsymbol{u}_b - \boldsymbol{u}_a)\hat{U}_{ab(1)},$$

self-collisions

"mass-ratio coefficients"

$$\hat{U}_{ab(1)} = \frac{3m_b(3T_a m_a + T_a m_b - 2T_b m_a)}{2(T_a m_b + T_b m_a)(m_a+m_b)};$$

$$\hat{D}_{ab(1)} = \{75T_a^3 m_a m_b^3 + 95T_a^3 m_b^4 + 174T_a^2T_b m_a m_b^3 + 300T_aT_b^2 m_a^3 m_b + 498T_aT_b^2 m_a^2 m_b^2 + 60T_b^3 m_a^4 + 104T_b^3 m_a^3 m_b\}\big[20(T_a m_b + T_b m_a)^3(m_a+m_b)\big]^{-1};$$

$$\hat{D}_{ab(2)} = \frac{9T_a m_b^2(10T_a^2 m_a m_b + 6T_a^2 m_b^2 + 45T_aT_b m_a^2 + 27T_aT_b m_a m_b - 14T_b^2 m_a^2)}{20(T_a m_b + T_b m_a)^3(m_a+m_b)};$$

$$\hat{E}_{ab(1)} = \frac{3T_a m_b(19T_a^2 m_a m_b^2 + 23T_a^2 m_b^3 - 2T_aT_b m_a^2 m_b + 36T_aT_b m_a m_b^2 + 84T_b^2 m_a^3 + 118T_b^2 m_a^2 m_b)}{560(T_a m_b + T_b m_a)^3(m_a+m_b)};$$

$$\hat{E}_{ab(2)} = \frac{9T_aT_b m_a m_b^2(7T_a m_a + 5T_a m_b - 2T_b m_a)}{112(T_a m_b + T_b m_a)^3(m_a+m_b)}.$$

4th-order moment exchange rates:

contains both viscosities
(but no heat fluxes)

$$\bar{\bar{\boldsymbol{Q}}}_a^{(4)\,\prime} = -\frac{53}{20}\nu_{aa}\frac{p_a}{\rho_a}\bar{\bar{\boldsymbol{\Pi}}}_a^{(2)} - \frac{79}{140}\nu_{aa}\bar{\bar{\boldsymbol{\Pi}}}_a^{(4)} + \sum_{b\neq a}\nu_{ab}\Big[-\hat{M}_{ab(1)}\frac{p_a}{\rho_a}\bar{\bar{\boldsymbol{\Pi}}}_a^{(2)} + \hat{M}_{ab(2)}\frac{p_a^2}{\rho_a p_b}\bar{\bar{\boldsymbol{\Pi}}}_b^{(2)} - N_{ab(1)}\bar{\bar{\boldsymbol{\Pi}}}_a^{(4)} - N_{ab(2)}\frac{p_a^2\rho_b}{p_b^2\rho_a}\bar{\bar{\boldsymbol{\Pi}}}_b^{(4)}\Big],$$

self-collisions

"mass-ratio coefficients"

$$\hat{M}_{ab(1)} = \Big\{48T_a^4 m_a m_b^3 + 36T_a^4 m_b^4 + 216T_a^3 T_b m_a^2 m_b^2 + 107T_a^3 T_b m_a m_b^3 + 378T_a^2 T_b^2 m_a^3 m_b + 36T_a^2 T_b^2 m_a^2 m_b^2 - 315T_a T_b^3 m_a^3 m_b - 70T_b^4 m_a^4\Big\}\Big[5(T_a m_b + T_b m_a)^3 T_a(m_b+m_a)\Big]^{-1};$$

$$\hat{M}_{ab(2)} = -\Big\{T_b m_a\big(18T_a^3 m_a m_b^2 - 4T_a^3 m_b^3 + 81T_a^2 T_b m_a^2 m_b - 18T_a^2 T_b m_a m_b^2 - 147T_a T_b^2 m_a^3 - 189T_a T_b^2 m_a^2 m_b + 35T_b^3 m_a^3\big)\Big\}\Big[5(T_a m_b + T_b m_a)^3 T_a(m_b+m_a)\Big]^{-1};$$

$$N_{ab(1)} = -\Big\{16T_a^3 m_a m_b^3 + 12T_a^3 m_b^4 + 72T_a^2 T_b m_a^2 m_b^2 + 21T_a^2 T_b m_a m_b^3 + 126T_a T_b^2 m_a^3 m_b - 54T_a T_b^2 m_a^2 m_b^2 - 140T_b^3 m_a^4 - 273T_b^3 m_a^3 m_b\Big\}\Big[35(T_a m_b + T_b m_a)^3 (m_b+m_a)\Big]^{-1};$$

$$N_{ab(2)} = -\frac{3T_b^2 m_a^2(35T_a^2 m_a m_b + 12T_a^2 m_b^2 - 35T_a T_b m_a^2 - 51T_a T_b m_a m_b + 7T_b^2 m_a^2)}{35(T_a m_b + T_b m_a)^3 T_a(m_b+m_a)}.$$

5th-order moment exchange rates:

$$\vec{\boldsymbol{Q}}_a^{(5)\,\prime} = -\Big[\frac{76}{5}\nu_{aa} + \sum_{b\neq a}\nu_{ab}\hat{F}_{ab(1)}\Big]\frac{p_a}{\rho_a}\vec{\boldsymbol{q}}_a + \sum_{b\neq a}\nu_{ab}\hat{F}_{ab(2)}\frac{p_a}{\rho_a}\frac{\rho_a}{\rho_b}\vec{\boldsymbol{q}}_b$$

$$-\Big[\frac{3}{35}\nu_{aa} + \sum_{b\neq a}\nu_{ab}\hat{G}_{ab(1)}\Big]\vec{\boldsymbol{X}}_a^{(5)} - \sum_{b\neq a}\nu_{ab}\hat{G}_{ab(2)}\frac{p_a}{p_b}\vec{\boldsymbol{X}}_b^{(5)}$$

$$-\frac{p_a^2}{\rho_a}\sum_{b\neq a}\nu_{ab}(\boldsymbol{u}_b - \boldsymbol{u}_a)\hat{U}_{ab(2)},$$

contains both heat fluxes (but no viscosities)

self-collisions

"mass-ratio coefficients"

$$\hat{U}_{ab(2)} = \frac{3m_b(17T_a^2 m_a m_b + 9T_a^2 m_b^2 + 42T_aT_b m_a^2 + 6T_aT_b m_a m_b - 28T_b^2 m_a^2)}{(T_a m_b + T_b m_a)^2(m_a+m_b)};$$

$$\hat{F}_{ab(1)} = \{855T_a^5 m_a m_b^4 + 759T_a^5 m_b^5 + 2340T_a^4T_b m_a^2 m_b^3 + 1972T_a^4T_b m_a m_b^4 + 2640T_a^3T_b^2 m_a^3 m_b^2$$
$$+2332T_a^3T_b^2 m_a^2 m_b^3 + 5880T_a^2T_b^3 m_a^4 m_b + 3324T_a^2T_b^3 m_a^3 m_b^2 - 3080T_aT_b^4 m_a^4 m_b - 560T_b^5 m_a^5\}$$
$$\times\big[10(T_a m_b + T_b m_a)^4(m_a+m_b)T_a\big]^{-1};$$

$$\hat{F}_{ab(2)} = 3T_a m_b^2\{70T_a^3 m_a m_b^2 + 102T_a^3 m_b^3 + 385T_a^2T_b m_a^2 m_b + 561T_a^2T_b m_a m_b^2 + 1890T_aT_b^2 m_a^3$$
$$+1446T_aT_b^2 m_a^2 m_b - 588T_b^3 m_a^3\}\big[10(T_a m_b + T_b m_a)^4(m_a+m_b)\big]^{-1};$$

$$\hat{G}_{ab(1)} = -\{565T_a^4 m_a m_b^4 + 533T_a^4 m_b^5 + 1270T_a^3T_b m_a^2 m_b^3 + 1190T_a^3T_b m_a m_b^4 + 1020T_a^2T_b^2 m_a^3 m_b^2$$
$$+1152T_a^2T_b^2 m_a^2 m_b^3 + 3640T_aT_b^3 m_a^4 m_b + 1916T_aT_b^3 m_a^3 m_b^2 - 1400T_b^4 m_a^5 - 3304T_b^4 m_a^4 m_b\}$$
$$\times\big[280(T_a m_b + T_b m_a)^4(m_a+m_b)\big]^{-1};$$

$$\hat{G}_{ab(2)} = -\frac{3T_aT_b m_a m_b^2(3T_a^2 m_a m_b - 5T_a^2 m_b^2 - 42T_aT_b m_a^2 - 38T_aT_b m_a m_b + 12T_b^2 m_a^2)}{8(T_a m_b + T_b m_a)^4(m_a+m_b)}.$$

The model is now fully specified, and represents **multi-fluid generalization of Braginskii**. Coupled with Maxwell's equations, it can be used in multi-fluid numerical simulations. Importantly, when collisional frequencies become small, the right-hand-sides just become small, and no coefficients "explode".
The Braginskii model is obtained as a particular case of the 1) one ion-electron plasma with similar temperatures, 2) applying quasi-static (no d/dt) and semi-linear approximation for Pi's & q's
3) where additionally, coupling between viscosity-tensors and heat fluxes is neglected.

Semi-linear approximation
(de-coupled viscosity-tensors and heat fluxes)

Instead of fully non-linear left-hand-sides of evolution equations (given 5 slides back), it is revealing to consider situations where **no large-scale gradients are present**, and simplify the equations by using the semi-linear approximation, where additionally the viscosity-tensors and heat fluxes are de-coupled. This removes many terms, including $(\nabla p_a)\vec{\boldsymbol{q}}_a,\ (\nabla p_a)\cdot\bar{\bar{\boldsymbol{\Pi}}}^{(2)}_a$
which might be significant if large-scale gradients in temperature/pressure are present.

The multi-fluid model is simplified with evolution equations for the heat fluxes:

$$\frac{d_a}{dt}\vec{\boldsymbol{q}}_a + \Omega_a\hat{\boldsymbol{b}}\times\vec{\boldsymbol{q}}_a + \frac{5}{2}p_a\nabla\Big(\frac{p_a}{\rho_a}\Big) = \vec{\boldsymbol{Q}}^{(3)}_a{}';$$

$$\frac{d_a}{dt}\vec{\boldsymbol{X}}^{(5)}_a + \Omega_a\hat{\boldsymbol{b}}\times\vec{\boldsymbol{X}}^{(5)}_a + 70\frac{p_a^2}{\rho_a}\nabla\Big(\frac{p_a}{\rho_a}\Big) = \vec{\boldsymbol{Q}}^{(5)}_a{}',$$

and viscosities:

$$\frac{d_a}{dt}\bar{\bar{\boldsymbol{\Pi}}}^{(2)}_a + \Omega_a\big(\hat{\boldsymbol{b}}\times\bar{\bar{\boldsymbol{\Pi}}}^{(2)}_a\big)^S + p_a\bar{\bar{\boldsymbol{W}}}_a = \bar{\bar{\boldsymbol{Q}}}^{(2)}_a{}';$$

$$\frac{d_a}{dt}\bar{\bar{\boldsymbol{\Pi}}}^{(4)}_a + \Omega_a\big(\hat{\boldsymbol{b}}\times\bar{\bar{\boldsymbol{\Pi}}}^{(4)}_a\big)^S + 7\frac{p_a^2}{\rho_a}\bar{\bar{\boldsymbol{W}}}_a = \bar{\bar{\boldsymbol{Q}}}^{(4)}_a{}'$$

The right-hand-sides are unchanged (given on previous 3 slides).
We will use this system to recover Braginskii.

One ion-electron plasma

(small temperature differences between ions and electrons)

ION species (let us consider only self-collisions, to be compatible with Braginskii)
First, let's explore reduction to the less-precise **13-moment model** of Burgers (1969)-Schunk (1977).
The ion heat flux evolves according to:

$$\frac{d_a}{dt}\vec{\boldsymbol{q}}_a + \Omega_a \hat{\boldsymbol{b}} \times \vec{\boldsymbol{q}}_a + \frac{5}{2} p_a \nabla\Big(\frac{p_a}{\rho_a}\Big) = -\frac{4}{5}\nu_{aa}\vec{\boldsymbol{q}}_a + \cancel{\frac{3}{70}\nu_{aa}\Big(\frac{\rho_a}{p_a}\vec{\boldsymbol{X}}^{(5)}_a - 28\vec{\boldsymbol{q}}_a\Big)}; \longleftarrow \text{ closure } \vec{\boldsymbol{X}}^{(5)}_a = 28\frac{p_a}{\rho_a}\vec{\boldsymbol{q}}_a;$$

$$\cancel{\frac{d_a}{dt}\vec{\boldsymbol{X}}^{(5)}_a + \Omega_a \hat{\boldsymbol{b}} \times \vec{\boldsymbol{X}}^{(5)}_a + 70\frac{p_a^2}{\rho_a}\nabla\Big(\frac{p_a}{\rho_a}\Big) = -\frac{88}{5}\nu_{aa}\frac{p_a}{\rho_a}\vec{\boldsymbol{q}}_a - \frac{3}{35}\nu_{aa}\Big(\vec{\boldsymbol{X}}^{(5)}_a - 28\frac{p_a}{\rho_a}\vec{\boldsymbol{q}}_a\Big)} \longleftarrow \text{ neglect}$$

- no problem if the collisional frequency \nu becomes small

Solution in a quasi-static approximation (cancel d/dt) yields ion heat flux:

$$\vec{\boldsymbol{q}}_a = -\kappa^a_\parallel \nabla_\parallel T_a - \kappa^a_\perp \nabla_\perp T_a + \kappa^a_\times \hat{\boldsymbol{b}} \times \nabla T_a,$$

with thermal conductivities:

Problem ! parallel cond. explodes with \nu->0

$$\kappa^a_\parallel = \frac{25}{8}\frac{p_a}{\nu_{aa} m_a};$$

$$\kappa^a_\perp = \frac{p_a}{m_a}\frac{2\nu_{aa}}{\Omega_a^2 + (4/5)^2\nu_{aa}^2};$$

$$\kappa^a_\times = \frac{p_a}{m_a}\frac{(5/2)\Omega_a}{\Omega_a^2 + (4/5)^2\nu_{aa}^2}.$$

rewritten with variable $x = \Omega_a/\nu_{aa}$

$$\kappa^a_\parallel = \underbrace{\frac{25}{8}}_{3.125}\frac{p_a}{\nu_{aa} m_a};$$

$$\kappa^a_\perp = \frac{p_a}{\nu_{aa} m_a}\frac{2}{x^2 + (4/5)^2};$$

$$\kappa^a_\times = \frac{p_a}{\nu_{aa} m_a}\frac{(5/2)x}{x^2 + (4/5)^2}.$$

Braginskii eq. (4.40),
higher-order polynomials in x, parallel different,
perp and cross match for x->infinity
(strong magnetic field)

$$\varkappa^i_\parallel = 3.906 n_i T_i \tau_i / m_i,$$

$$\varkappa^i_\perp = (n_i T_i \tau_i/m_i)(2x^2 + 2.645)/\Delta,$$

$$\varkappa^i_\wedge = (n_i T_i \tau_i/m_i)\, x\left(\frac{5}{2}x^2 + 4.65\right)/\Delta,$$

$$\Delta = x^4 + 2.70x^2 + 0.677.$$

Ion heat flux of Braginskii (self-collisions)

The explosion of the parallel heat flux in a weakly-collisional regime is obviously a problem of the quasi-static approximation, and (in the homogeneous regime considered here) the evolution equations should be retained.

Analogous (but more precise) situation, **evolution equations of our 21-moment model**:

$$\frac{d_a}{dt}\vec{\boldsymbol{q}}_a + \Omega_a \hat{\boldsymbol{b}} \times \vec{\boldsymbol{q}}_a + \frac{5}{2} p_a \nabla\Big(\frac{p_a}{\rho_a}\Big) = -2\nu_{aa}\vec{\boldsymbol{q}}_a + \frac{3}{70}\nu_{aa}\frac{\rho_a}{p_a}\vec{\boldsymbol{X}}^{(5)}_a;$$

$$\frac{d_a}{dt}\vec{\boldsymbol{X}}^{(5)}_a + \Omega_a \hat{\boldsymbol{b}} \times \vec{\boldsymbol{X}}^{(5)}_a + 70\frac{p_a^2}{\rho_a}\nabla\Big(\frac{p_a}{\rho_a}\Big) = -\frac{76}{5}\nu_{aa}\frac{p_a}{\rho_a}\vec{\boldsymbol{q}}_a - \frac{3}{35}\nu_{aa}\vec{\boldsymbol{X}}^{(5)}_a$$

- no problem in the corona, the r.h.s. just become small

Quasi-static solution yields the same form for the ion heat flux:

$$\vec{\boldsymbol{q}}_a = -\kappa^a_{\parallel}\nabla_{\parallel}T_a - \kappa^a_{\perp}\nabla_{\perp}T_a + \kappa^a_{\times}\hat{\boldsymbol{b}}\times\nabla T_a,$$

(similar solution for X^5 - not shown)

but now with **fully analytic ion thermal conductivities:**

$$\kappa^a_{\parallel} = \frac{125}{32}\frac{p_a}{\nu_{aa}m_a};$$

$$\kappa^a_{\perp} = \frac{p_a}{\nu_{aa}m_a}\frac{2x^2+(648/245)}{x^4+(3313/1225)x^2+(20736/30625)};$$

$$\kappa^a_{\times} = \frac{p_a}{\nu_{aa}m_a}\frac{(5/2)x^3+(2277/490)x}{x^4+(3313/1225)x^2+(20736/30625)},$$

or with **numerical values**, recovering the ion heat flux

$$\kappa^a_{\parallel} = 3.906\frac{p_a}{\nu_{aa}m_a};$$

$$\kappa^a_{\perp} = \frac{p_a}{\nu_{aa}m_a}\frac{2x^2+2.645}{x^4+2.704x^2+0.6771};$$

$$\kappa^a_{\times} = \frac{p_a}{\nu_{aa}m_a}\frac{(5/2)x^3+4.647x}{x^4+2.704x^2+0.6771},$$

Braginskii eq. (4.40)

$$\varkappa^i_{\parallel} = 3.906 n_i T_i \tau_i/m_i,$$

$$\varkappa^i_{\perp} = (n_i T_i \tau_i/m_i)\,(2x^2+2.645)/\Delta,$$

$$\varkappa^i_{\wedge} = (n_i T_i \tau_i/m_i)\,x\Big(\frac{5}{2}x^2+4.65\Big)/\Delta,$$

$$\Delta = x^4+2.70x^2+0.677.$$

=> Formulation with the above evolution equations is a really powerful tool for clarifying the Braginskii model, the evolution equations are so simple ! (Perhaps this is how the model should be thought at universities).

Viscosity-tensors are also described by coupled evolution equations

Ion viscosity of Braginskii (self-collisions)

For the **21-moment model**, the evolution equations are:

$$\frac{d_a}{dt}\bar{\bar{\Pi}}_a^{(2)} + \Omega_a\big(\hat{\boldsymbol{b}}\times\bar{\bar{\Pi}}_a^{(2)}\big)^S + p_a\bar{\bar{\boldsymbol{W}}}_a = -\frac{21}{10}\nu_{aa}\bar{\bar{\Pi}}_a^{(2)} + \frac{9}{70}\nu_{aa}\frac{\rho_a}{p_a}\bar{\bar{\Pi}}_a^{(4)};$$

$$\frac{d_a}{dt}\bar{\bar{\Pi}}_a^{(4)} + \Omega_a\big(\hat{\boldsymbol{b}}\times\bar{\bar{\Pi}}_a^{(4)}\big)^S + 7\frac{p_a^2}{\rho_a}\bar{\bar{\boldsymbol{W}}}_a = -\frac{53}{20}\nu_{aa}\frac{p_a}{\rho_a}\bar{\bar{\Pi}}_a^{(2)} - \frac{79}{140}\nu_{aa}\bar{\bar{\Pi}}_a^{(4)}.$$

- no explosion in corona, r.h.sides just become small

quasi-static solution, equivalent to Braginskii (4.41) & (4.42):

$$\bar{\bar{\Pi}}_a^{(2)} = -\eta_0^a\bar{\bar{\boldsymbol{W}}}_0 - \eta_1^a\bar{\bar{\boldsymbol{W}}}_1 - \eta_2^a\bar{\bar{\boldsymbol{W}}}_2 + \eta_3^a\bar{\bar{\boldsymbol{W}}}_3 + \eta_4^a\bar{\bar{\boldsymbol{W}}}_4;$$

$$\bar{\bar{\boldsymbol{W}}}_0 = \frac{3}{2}(\bar{\bar{\boldsymbol{W}}}_a : \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\Big(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\bar{\bar{\boldsymbol{I}}}}{3}\Big);$$

$$\bar{\bar{\boldsymbol{W}}}_1 = \bar{\bar{\boldsymbol{I}}}_\perp\cdot\bar{\bar{\boldsymbol{W}}}_a\cdot\bar{\bar{\boldsymbol{I}}}_\perp + \frac{1}{2}(\bar{\bar{\boldsymbol{W}}}_a : \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\bar{\bar{\boldsymbol{I}}}_\perp;$$

$$\bar{\bar{\boldsymbol{W}}}_2 = (\bar{\bar{\boldsymbol{I}}}_\perp\cdot\bar{\bar{\boldsymbol{W}}}_a\cdot\hat{\boldsymbol{b}}\hat{\boldsymbol{b}})^S;$$

$$\bar{\bar{\boldsymbol{W}}}_3 = \frac{1}{2}(\hat{\boldsymbol{b}}\times\bar{\bar{\boldsymbol{W}}}_a\cdot\bar{\bar{\boldsymbol{I}}}_\perp)^S;$$

$$\bar{\bar{\boldsymbol{W}}}_4 = (\hat{\boldsymbol{b}}\times\bar{\bar{\boldsymbol{W}}}_a\cdot\hat{\boldsymbol{b}}\hat{\boldsymbol{b}})^S,$$

(see e.g. Appendix E.4 of Hunana et al. 2022)

rate-of-strain tensor

$$\bar{\bar{\boldsymbol{W}}}_a = (\nabla\boldsymbol{u}_a)^S - (2/3)\bar{\bar{\boldsymbol{I}}}\nabla\cdot\boldsymbol{u}_a.$$

$$\eta_1^a(x) = \eta_2^a(2x);$$
$$\eta_3^a(x) = \eta_4^a(2x)$$

ion viscosities:

$$\eta_0^a = \frac{1025}{1068}\frac{p_a}{\nu_{aa}};$$

parallel viscosity explodes in corona

$$\eta_2^a = \frac{p_a}{\nu_{aa}}\frac{(6/5)x^2 + (10947/4900)}{x^4 + (79321/19600)x^2 + (71289/30625)};$$

$$\eta_4^a = \frac{p_a}{\nu_{aa}}\frac{x^3 + (46561/19600)x}{x^4 + (79321/19600)x^2 + (71289/30625)},$$

$$x = \Omega_a/\nu_{aa}$$

numerical values:

$$\eta_0^a = 0.960\frac{p_a}{\nu_{aa}};$$

$$\eta_2^a = \frac{p_a}{\nu_{aa}}\frac{(6/5)x^2 + 2.234}{x^4 + 4.047x^2 + 2.328};$$

$$\eta_4^a = \frac{p_a}{\nu_{aa}}\frac{x^3 + 2.376x}{x^4 + 4.047x^2 + 2.328},$$

(similar solution for Pi^4 - not shown)

Braginskii eq. (4.44):

$$\eta_0^i = 0.96 n_i T_i \tau_i,$$

$$\eta_2^i = n_i T_i \tau_i\left(\frac{6}{5}x^2 + 2.23\right)/\Delta,$$

$$\eta_4^i = n_i T_i \tau_i x(x^2 + 2.38)/\Delta,$$

$$\Delta = x^4 + 4.03x^2 + 2.33.$$

one of his values is slightly imprecise, should be 4.05 instead

Discussion and conclusions

Energy conservation

The collisional contributions were calculated in the semi-linear approximation, where quantities such as $\vec{q}_a \cdot (\boldsymbol{u}_b - \boldsymbol{u}_a)$ and $|\boldsymbol{u}_b - \boldsymbol{u}_a|^2$ are neglected and considered small.

Thus, an exact energy conservation $Q_{ab} + Q_{ba} = (\boldsymbol{u}_b - \boldsymbol{u}_a) \cdot \boldsymbol{R}_{ab}$ cannot be achieved, because the collisional integrals would have to be calculated fully non-linearly. In the multi-fluid description, the energy is conserved only approximately. This is consistent with the models of Burgers-Schunk. An exact energy conservation can be only achieved in two particular cases:

1) A one ion-electron plasma (or a two-species plasma)

$$Q_{ie} = 3n_e\nu_{ei}(T_e - T_i)\frac{m_e}{m_i}; \qquad Q_{ei} = -Q_{ie} - (\boldsymbol{u}_e - \boldsymbol{u}_i) \cdot \boldsymbol{R}_{ei},$$

define/calculate

Impose by hand. This is the choice of Braginskii.

2) Completely neglecting the heat fluxes and stress-tensors (i.e. considering only the 5-moment model). Calculations can be actually done for unrestricted drifts (Dreicer, Burgers, Schunk, Tanenbaum, Balescu)

$$\boldsymbol{R}_{ab} = \rho_a\nu_{ab}(\boldsymbol{u}_b - \boldsymbol{u}_a)\Phi_{ab};$$

$$Q_{ab} = \rho_a\nu_{ab}\Big[3\frac{T_b - T_a}{m_a + m_b}\Psi_{ab} + \frac{m_b}{m_a + m_b}|\boldsymbol{u}_b - \boldsymbol{u}_a|^2\Phi_{ab}\Big],$$

$$\Phi_{ab} = \Big(\frac{3}{4}\sqrt{\pi}\frac{\mathrm{erf}(\epsilon)}{\epsilon^3} - \frac{3}{2}\frac{e^{-\epsilon^2}}{\epsilon^2}\Big);$$

$$\Psi_{ab} = e^{-\epsilon^2}; \qquad \epsilon = \frac{|\boldsymbol{u}_b - \boldsymbol{u}_a|}{\sqrt{v_{\mathrm{th}a}^2 + v_{\mathrm{th}b}^2}},$$

Many things that were not discussed here (see our paper):

- Analytic solutions for the electron species (necessary to consider both electron-electron and electron-ion collisions).

- Analytic solutions where the coupling between stress-tensors and heat fluxes is accounted for, i.e. when a stress-tensor enters a heat flux, and a heat flux enters a stress-tensor.

- 22-moment model, where one accounts for a fully contracted scalar perturbation of the 4th-order fluid moment (can be viewed as reduced kurtosis, describing if the distribution function is tail-heavy or tail light), which modify the energy exchange rates.

- **Limitations of our approach in a weakly-collisional regime:**

    - Necessity to expand around anisotropic bi-Maxwellian distribution instead of Maxwellian.

    - **Our Braginskii-type models do not contain the effect of Landau damping** (non-local thermal heat fluxes) nor do we have the free-streaming heat fluxes of Hollweg (1974, 1976).

    - Positivity of the perturbed distribution function (clear in the homogeneous case - need to retain d/dt; not so clear when large-scale gradients are present).

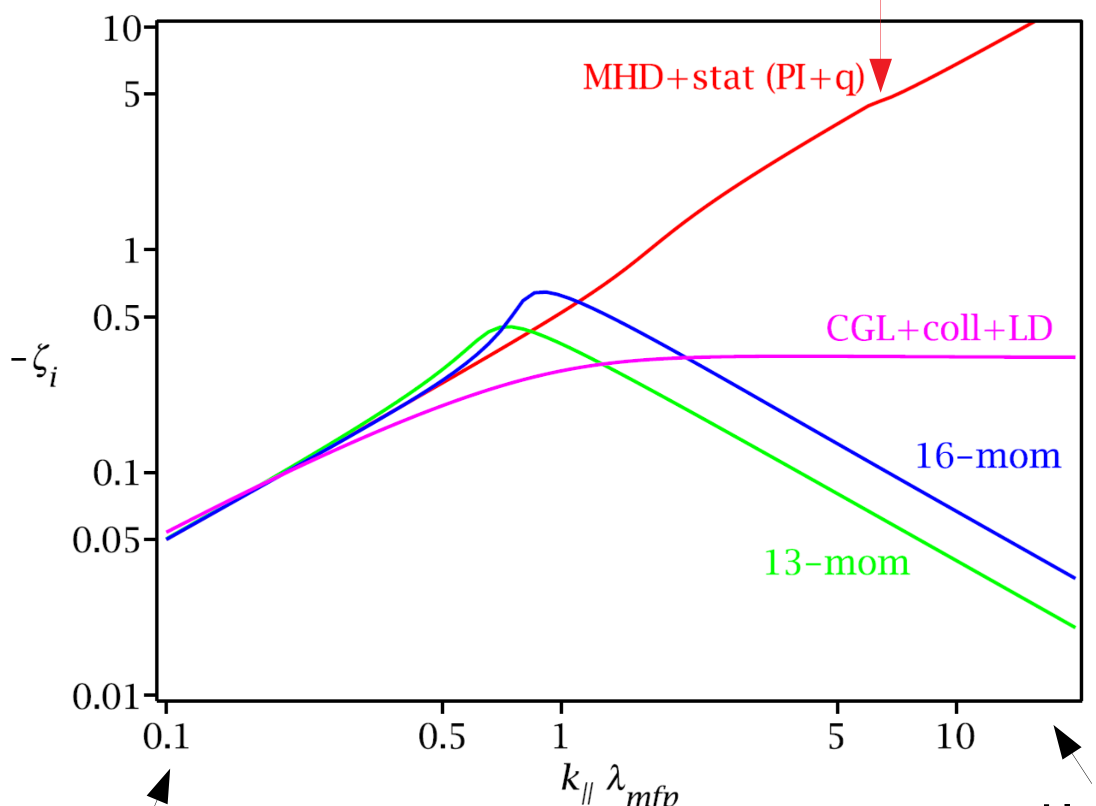
Importance of Landau damping

Let us compare **damping of** the parallel propagating **sound** (ion-acoustic) mode in a homogeneous medium. We compare only simple models, proton-electron plasma with cold electrons (which is not well-defined kinetically). The goal is to describe general behavior and not to provide precise damping rates.

~~d/dt~~ damping increases and the mode stops existing (consequence of static Pi, static q just yields an isothermal behavior, with no damping present)

imaginary part (damping) of normalized phase speed

$$\zeta = \frac{\omega}{|k_{||}| v_{\mathrm{th}||}}$$



Landau damping (closures of Snyder et al. 1997) converges to a constant.

anisotropic, d/dt retained,
isotropic, d/dt retained,
both reach maximum and decrease to zero.

highly-collisional

wavenumber * mean-free-path

weakly-collisional

While a vanishing damping is preferred against a quantity that blows up in a weakly-collisional regime, all 3 classical models are technically incorrect, because LD provides significant contribution to the damping rate. Braginskii-type models provide reasonable prediction if the wavelength is larger than the mean-free path.

Examples where our model might be useful

One ion-electron plasma: Instead of just measuring the usual heat flux in the solar wind observational studies (and kinetic simulations), we propose to also directly measure:

$$\vec{\boldsymbol{q}}_e = \frac{\vec{\boldsymbol{X}}_e^{(3)}}{2} = -3.2\frac{p_e}{m_e\nu_{ei}}\nabla T_e; \qquad \vec{\boldsymbol{X}}_e^{(5)} = -110.7\frac{p_e^2}{\rho_e m_e \nu_{ei}}\nabla T_e, \qquad \widetilde{X}_e^{(4)} = +83.8\frac{p_e^2}{\nu_{ei}^2\rho_e m_e}\nabla^2 T_e.$$

Spitzer-Braginskii

our model

Multi-fluid: Modeling of enrichment of minor ion abundancies in stellar atmospheres, because of the very precise thermal force (thermal diffusion).

3 major multi-fluid models (for simplicity let us only compare thermal forces given by):

$$\text{Burgers-Schunk:} \qquad \boldsymbol{R}_e^T = +\frac{3}{5}\frac{\rho_e}{p_e}\nu_{ei}\vec{\boldsymbol{q}}_e;$$

$$\text{Killie et al.:} \qquad \boldsymbol{R}_e^T = +\frac{6}{35}\frac{\rho_e}{p_e}\nu_{ei}\vec{\boldsymbol{q}}_e;$$

$$\text{present paper:} \qquad \boldsymbol{R}_e^T = +\frac{21}{10}\frac{\rho_e}{p_e}\nu_{ei}\vec{\boldsymbol{q}}_e - \frac{3}{56}\frac{\rho_e^2}{p_e^2}\nu_{ei}\vec{\boldsymbol{X}}_e^{(5)}.$$

ion charge Z_i = 1:

	heat conductivity γ_0	thermal force β_0
Burgers-Schunk	1.34	0.804
Killie et al.	3.92	0.672
Our model (=Braginskii)	3.1616	0.711
Spitzer-Härm	3.203	0.703

A problem arises if one uses the correct thermal conductivity (from observation data) to determine the thermal force

if correct γ_0	thermal force β_0
$3.2 \rightarrow$	1.92
$3.2 \rightarrow$	0.55

(Additionally, the model of Killie et al. breaks the Onsager symmetry)

=> We offer an improvement to the multi-fluid models of Burgers-Schunk and Killie et al.

For a review of Braginskii-type models and its generalizations see:

The Astrophysical Journal Supplement Series, 260:26 (145pp), 2022 June

https://arxiv.org/abs/2201.11561

Generalized Fluid Models of the Braginskii Type

P. Hunana[1,2], T. Passot[3], E. Khomenko[1,2], D. Martínez-Gómez[1,2], M. Collados[1,2], A. Tenerani[4], G. P. Zank[5,6], Y. Maneva[7], M. L. Goldstein[8], and G. M. Webb[5]

For a review of collisionless (bi-Maxwellian) CGL-type models see:

J. Plasma Phys. (2019), *vol.* 85, 205850602

https://arxiv.org/abs/1901.09354

An introductory guide to fluid models with anisotropic temperatures. Part 1. CGL description and collisionless fluid hierarchy

P. Hunana[1,2,†], A. Tenerani[3], G. P. Zank[4,5], E. Khomenko[1,2], M. L. Goldstein[6], G. M. Webb[4], P. S. Cally[7], M. Collados[1,2], M. Velli[8] and L. Adhikari[4]

For a review of Landau fluid models see:

J. Plasma Phys. (2019), *vol.* 85, 205850603

https://arxiv.org/abs/1901.09360

An introductory guide to fluid models with anisotropic temperatures. Part 2. Kinetic theory, Padé approximants and Landau fluid closures

P. Hunana[1,2,†], A. Tenerani[3], G. P. Zank[4,5], M. L. Goldstein[6], G. M. Webb[4], E. Khomenko[1,2], M. Collados[1,2], P. S. Cally[7], L. Adhikari[4] and M. Velli[8]