Some general results on relative magnetic helicity and field line helicity

Jean-Jacques Aly

Department of Astrophysics, CE Saclay, France

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Abstract

- We present some general considerations on two quantities that are of common use in solar physics: the ***relative magnetic helicity*** H and the ***field line helicity*** h of a magnetic field **B** contained in some domain D.
- We show how these two quantities can be expressed in terms of either the magnetic mapping of **B** or, when **B** has a simple topology, the boundary values of two pairs of Euler potentials. The well-known topological invariance of H and h can be immediately seen on the formulae that are presented.
- We compute how the field line helicity varies in time when the plasma in D has finite resistivity and the footpoints of the magnetic lines on the boundary of D are submitted to shearing motions.

1. Definitions

- Notations and assumptions:
  - D = simply connected domain of space bounded by the connected surface S. **n** = outer normal to S.
  - **B** = smooth magnetic field contained in D. We assume that (almost) all the magnetic lines of **B** have two footpoints on S.
  - S^+, S^-, and S^0, denote the parts of S where $-B_n > 0$ (positive polarity), $-B_n < 0$ (negative polarity), and $B_n = 0$, respectively. S^0 is assumed here to be a curve (polarity inversion line, PIL), possibly made of several pieces.
  - $\mathcal{L}(\mathbf{r})$ = field line of **B** entering D at $\mathbf{r} \in S^+$. $\mathcal{L}(\mathbf{r})$ exits D at the point **M** = **M**(**r**) of S^-. The mapping **M**: $S^+ \rightarrow S^-$ so defined is called the ***magnetic mapping*** of **B**.

- Select:
  - a ***reference field*** $\mathbf{B}_r$ in D having the same normal component as **B** on S ($B_{rn} = B_n$);
  - a ***reference vector potential*** $\mathbf{A}_r$ of $\mathbf{B}_r$ ($\mathbf{B}_r = \nabla \times \mathbf{A}_r$).
- Let **A** by an arbitrary vector potential of **B** ($\mathbf{B} = \nabla \times \mathbf{A}$). Then the ***magnetic helicity of B relative to*** $\boldsymbol{B_r}$ is defined by (Berger & Field 1984, Finn and Antonsen 1985)

$$\mathrm{H}[\mathbf{B}/\mathbf{B}_r] = \int_D (\mathbf{A} + \mathbf{A}_r).(\mathbf{B} - \mathbf{B}_r)\mathrm{dv}\,.$$

- H is a gauge invariant quantity: it does not depend on the choices of **A** and $\mathbf{A}_r$.

- Most often, $\mathbf{B}_r$ chosen to be the unique potential field $\mathbf{B}_\pi$ satisfying $B_{\pi n} = B_n$ on S. In that case, the helicity of **B** w.r.t. $\mathbf{B}_\pi$ depends only on **B** – it is then an intrinsic property of that field – and one set

$$H_{rel}[\mathbf{B}] = H[\mathbf{B}/\mathbf{B}_\pi].$$

$H_{rel}[\mathbf{B}]$ is simply called the ***relative helicity*** of **B**.

- Impose the gauge condition (**gc**, hereafter)

$$\mathbf{A}\times\mathbf{n} = \mathbf{A}_r\times\mathbf{n} \qquad \text{on S.}$$

Then the ***field line helicity of*** **B** ***relative to*** $\mathbf{A}_r$ is the function defined on S^+ by (Berger 1988)

$$h[\mathbf{B}/\mathbf{A}_r;\mathbf{r}] = \int_{\mathcal{L}(\mathbf{r})} \mathbf{A}\cdot d\mathbf{l}.$$

- h is invariant under the gauge transforms of **A** respecting **gc** (this justifies the notation $h(\mathbf{B}/\mathbf{A}_r;\mathbf{r})$).
- H and h are related by

$$H[\mathbf{B}/\mathbf{B}_r] = \int_{S^+} h[\mathbf{B}/\mathbf{A}_r]\,(-B_n)ds - H_r\,,$$

$$H_r = \int_D \mathbf{A}_r.\mathbf{B}_r dv.$$

- Essential property of H and h: if $\mathbf{B}_1$ and $\mathbf{B}_2$ have the same topology – meaning here that they can be deformed into each other by ideal MHD motions keeping fixed the positions of the footpoints on S (which implies that $B_{1n}=B_{2n}$ on S) – then

$H[\mathbf{B}_1/\mathbf{B}_r] = H[\mathbf{B}_2/\mathbf{B}_r]$ and $h[\mathbf{B}_1/\mathbf{A}_r;\mathbf{r}] = h[\mathbf{B}_2/\mathbf{A}_r;\mathbf{r}]$.

2. Topology of B

- The field **B** is said to have a ***simple topology*** if its magnetic mapping **M** is continuous.
- In the opposite case, **B** has ***complex topology***. Generically, **M** is discontinuous across some arcs $\Gamma_j \subset S^+$: if $\mathbf{r}_1$ and $\mathbf{r}_2$ are located on either side of Γ_j, $\mathbf{M}(\mathbf{r}_1)$ and $\mathbf{M}(\mathbf{r}_2)$ are separated by a finite distance.
- The magnetic lines connected to Γ_j form a singular surface in D, a so-called ***separatrix***, which either contains a ***neutral point*** of **B** (where **B** = 0) or is tangent to S along a so-called ***bald patch*** $\subset S^0$.
- The domain $S^+\backslash(\Gamma_1 \cup \Gamma_2 \cup \ldots)$ decomposes into N cells S^+_k inside which **M** is continuous. We set $S^-_k = \mathbf{M}(S^+_k) \subset S^-$.

3. Expressing h and H in terms of **M**

A. Computation of h

- Fix a base point $\mathbf{r}_k$ in S^+_k. Then one gets by applying Stokes theorem to an adequately chosen magnetic surface (see also Aly 2014, Yeates & Hornig 2014)

$$\mathrm{h}(\mathbf{r}) = \mathrm{h}_k + \chi_k(\mathbf{r}), \qquad \chi_k(\mathbf{r}) = -\int_{\mathcal{C}(\mathbf{r}_k,\mathbf{r})} (\mathbf{A}_{\mathrm{rs}} - \nabla_s \mathbf{M} \cdot \widetilde{\mathbf{A}_{\mathrm{rs}}}) \cdot \mathrm{d}\mathbf{l}\,,$$

where

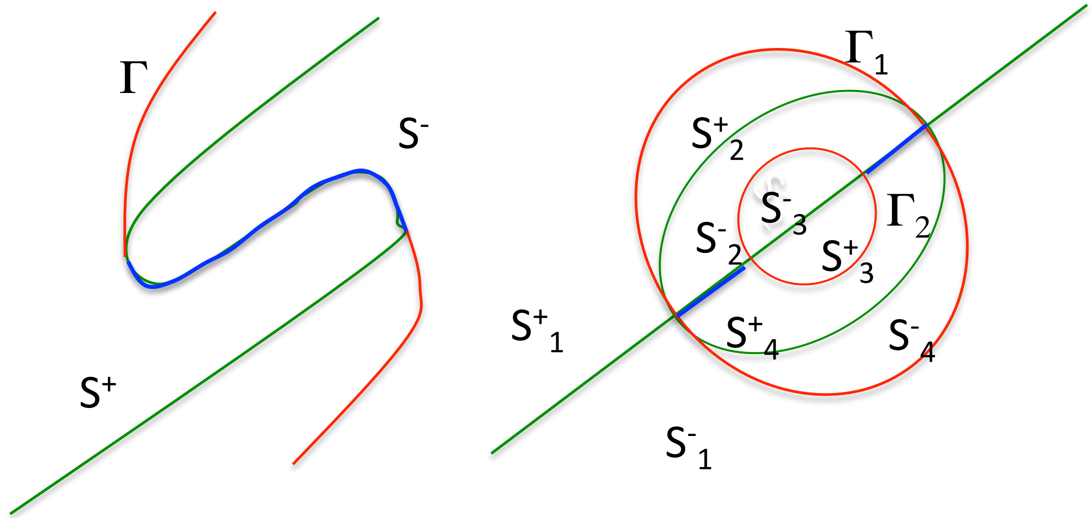
- $\mathbf{r}$ is an arbitrary point of S^+_k and $h_k = h(\mathbf{r}_k)$;
- $\mathcal{C}(\mathbf{r}_k, \mathbf{r})$ is an arbitrary curve connecting $\mathbf{r}_k$ to $\mathbf{r}$ on S^+_k;
- $\widetilde{X}(\mathbf{r}) = X(\mathbf{M}(\mathbf{r}))$ and $\mathbf{X}_s$ = component of $\mathbf{X}$ parallel to S.

- When ∂S^+_k and ∂S^-_k have a common part $\partial_k \subset S^0$ over which the lines are bridging, we can choose $\mathbf{r}_k$ on ∂_k. Then $h_k = 0$ and h is fully determined in S^+_k by

$$h(\mathbf{r}) = \chi_k(\mathbf{r}) \ .$$

- This happens for instance:
  - When **B** has a simple topology (in which case N=1, $S^+_1 = S^+$, $S^-_1 = S^-$, $\partial_1 = S^0$).
  - For adequate choices of the functions m, n, and p in the following model: in each plane x=const, **B** coincides with the field created by two 2D dipoles, one of moment $m(x)\mathbf{e}_y$ located at (y=-d,z=-p(x)), and one of moment $n(x)\mathbf{e}_y$ located at (y=d,z=-p(x)). One may then get configurations with the following structures on S:

Green: PIL S^0
Blue: bald patch
Red: trace of a separatrix



Obviously, both structures allow choosing all the $\mathbf{r}_k$ on S^0.

- If this is not the case, we choose $\mathbf{r}_k$ on a discontinuity curve Γ_j of **M** and we compute the constant h_k by using the continuity of **A** and **B** on the separatrices. A simple example is as follows.

- Consider an axisymmetric quadrupolar field in the exterior of a spherical domain. We first impose $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, to lie on a polarity inversion line, whence

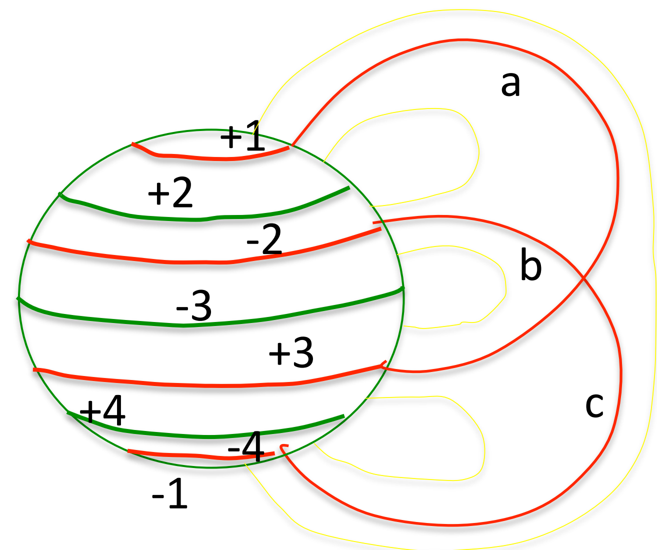
$$h_2=h_3=h_4=0\ .$$

Next we choose $\mathbf{r}_1$ on the separatrix (in red), and note that

$$h_1=h_a+h_c-h_b\ ,$$

where h_a, h_b, h_c can be computed inside the regions 2,3,4 with help from the relation

$$h(\mathbf{r}) = \chi_k(\mathbf{r}) \quad \text{in } S^+_k.$$



B. Helicity

- Using the expression above for h and the relation between h and H, we obtain

$$H + H_r = \sum_{k=1}^{N}\left[h_k \phi_k + \int_{S_k^+} \chi_k (-B_n)\, ds\right]$$

$$= \sum_{k=1}^{N}\left[h_k \phi_k + \int_{S_k^+} (\nabla_s \mathbf{M}.\widetilde{\mathbf{A}_{rs}} \times \mathbf{A}_{rs}).\mathbf{n} ds - \int_{\partial S_k^+} \chi_k \mathbf{A}_{rs}.\, d\mathbf{l}\right]$$

where Φ_k = magnetic flux through S^+_k ($\Phi_k > 0$).

- The formulae obtained for h and H clearly exhibit the topological invariance of these quantities.

- When **B** has a simple topology, one gets (Aly 2018)

$$\mathrm{H} = \int_{S^+} [(\boldsymbol{\nabla_s}\mathbf{M}.\widetilde{\mathbf{A}_{\mathrm{rs}}})\times\mathbf{A}_{\mathrm{rs}}]\cdot\mathbf{n}\,\mathrm{ds} - \mathrm{H}_r$$
$$= \int \varepsilon^{kj} A_{rj}\widetilde{A_{r\iota}}\,\partial_k X^i dx^1 dx^2 - \mathrm{H}_r$$
$$= \int_{S^+} [(\boldsymbol{\nabla_s}\mathbf{M}.\widetilde{\mathbf{A}_{\mathrm{rs}}} - \boldsymbol{\nabla_s}\mathbf{M}_r.\widetilde{\mathbf{A}_{\mathrm{rs}}}^{r})\times\mathbf{A}_{\mathrm{rs}}]\cdot\mathbf{n}\,\mathrm{ds}.$$

- In the second line, we have used coordinates (x^1,x^2) on S^+ and (X^1,X^2) on S^-, and expressed the magnetic mapping as

$$\mathbf{M}: (x^1,x^2) \mapsto (X^1(x^1,x^2),X^2(x^1,x^2)).$$

ε^{kj} denotes the 2D alternating tensor.

- Third line valid if $\mathbf{B}_r$, too, has a simple topology; $\mathbf{M}_r$ = magnetic mapping of $\mathbf{B}_r$; $\widetilde{\mathbf{A}_{\mathrm{rs}}}^{r}(\mathbf{r}) = \mathbf{A}_{\mathrm{rs}}(\mathbf{M}_r(\mathbf{r}))$.

4. Helicity and Euler potentials

- Assume that both **B** and $\mathbf{B}_r$ have simple topology.
- For such fields, one can introduce the ***global Euler representations*** (Aly 1990, 2018)

$$\mathbf{B} = \nabla U \times \nabla V \quad \text{and} \quad \mathbf{B}_r = \nabla U_r \times \nabla V_r,$$

with $U = U_r$ and $V = V_r$ on S^+ and all the level contours of V_r on S^+ cutting ∂S^+. Clearly, one has

$$(U,V)(\mathbf{M}(\mathbf{r})) = (U,V)(\mathbf{r}) \quad \text{for } \mathbf{r} \in S^+.$$

- Note that one can write for any **A** and $\mathbf{A}_r$

$$\mathbf{A} = U \nabla V + \nabla f \quad \text{and} \quad \mathbf{A}_r = U_r \nabla V_r + \nabla f_r$$

for some functions f and f_r.

- Then one has for the line helicity

$$h[\mathbf{B}/\mathbf{A}_r;\mathbf{r}]=f(\mathbf{M}(\mathbf{r}))-f(\mathbf{r}),$$

with

  - $f(\mathbf{r}) = f_r(\mathbf{r})$ $\quad\quad \mathbf{r}\in S^+,$
  - $f(\mathbf{r}) = f_r(\mathbf{r}) + \int_{C(\mathbf{r}_1,\mathbf{r})}(U_r\nabla V_r - U\nabla V).\,d\mathbf{l}, \quad \mathbf{r}\in S^-.$

- For the helicity (Aly 1990, 2018), one gets

$$H[\mathbf{B}/\mathbf{B}_r] = \int_{S^-} UU_r(\nabla_s V\times\nabla_s V_r)\cdot\mathbf{n}\,ds.$$

- Again, we have formulae clearly exhibiting the topological invariance of h and H as **M** and (U,V) on S are unchanged when **B** is deformed.

5. A formula for the evolution of h

- We consider here a simple situation defined by the following assumptions:
  - **B**(**r**,t) evolves by keeping a simple topology.
  - This evolution is driven by:
    - Tangential motions imposed to the plasma on the perfectly conducting boundary S. These motions conserve B_n, and then there is some function ζ such that

$$\mathbf{v}_s = \mathbf{n} \times \nabla_s \zeta \; / \; B_n \; .$$

    - Non-ideal MHD processes acting in D and described by the term **N** in Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B}/c = \mathbf{N} \; .$$

N is taken to vanish in a neighborhood of S.

- As B_n is preserved, we can select a reference field $\mathbf{B}_r$, a reference vector potential $\mathbf{A}_r$, and Euler potentials U_r and V_r that are all time-independent.
- We choose $\mathbf{A}_r$ to be of the form

$$\mathbf{A}_r = U_r \nabla V_r \, .$$

- We consider a magnetic line $\mathcal{L}(t)$ which is attached to a given element of matter located at $\mathbf{r}(t)$ on S^+ and whose footpoint on S^+ thus moves at the velocity $\mathbf{v}_s$.
- Our aim is to compute the time derivative of the quantity

$$h(t) = h[\mathbf{B}(t)/\mathbf{A}_r \, ; \mathbf{r}(t)].$$

- Some formulae for dh/dt have previously been given by Russell, Yeates, Hornig & Wilmot-Smith (2015).

- One gets after some algebra (Aly 2018)

$$\frac{dh}{dt}(t) = \left[U_r \frac{\partial(\zeta + \mathcal{N})}{\partial U_r} - (\zeta + \mathcal{N}) \right]_{\mathcal{L}(t)},$$

where $[X]_{\mathcal{L}} = X[\mathbf{M}(\mathbf{r})] - X[\mathbf{r}]$,

$$\mathcal{N} = \int_{\mathcal{L}} c\mathbf{N} \,.\, d\mathbf{l} \,,$$

and we have used (U_r, V_r) as coordinates on S.

- This formula can be generalized to the case where:
  - The boundary motions do not preserve B_n, with the velocity thus being of the general form $\mathbf{v}_s = (\mathbf{n} \times \nabla_s \zeta + \nabla_s \theta) / B_n$.
  - The reference vector potential $\mathbf{A}_r$ is time-dependent and not necessarily of the form $\mathbf{A}_r = U_r \nabla V_r$.

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