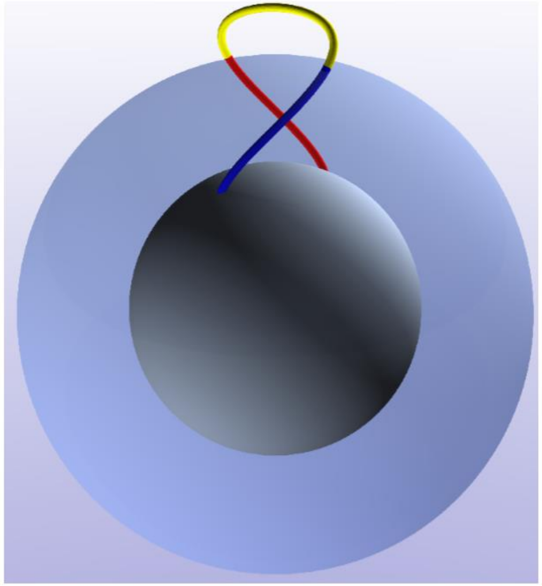
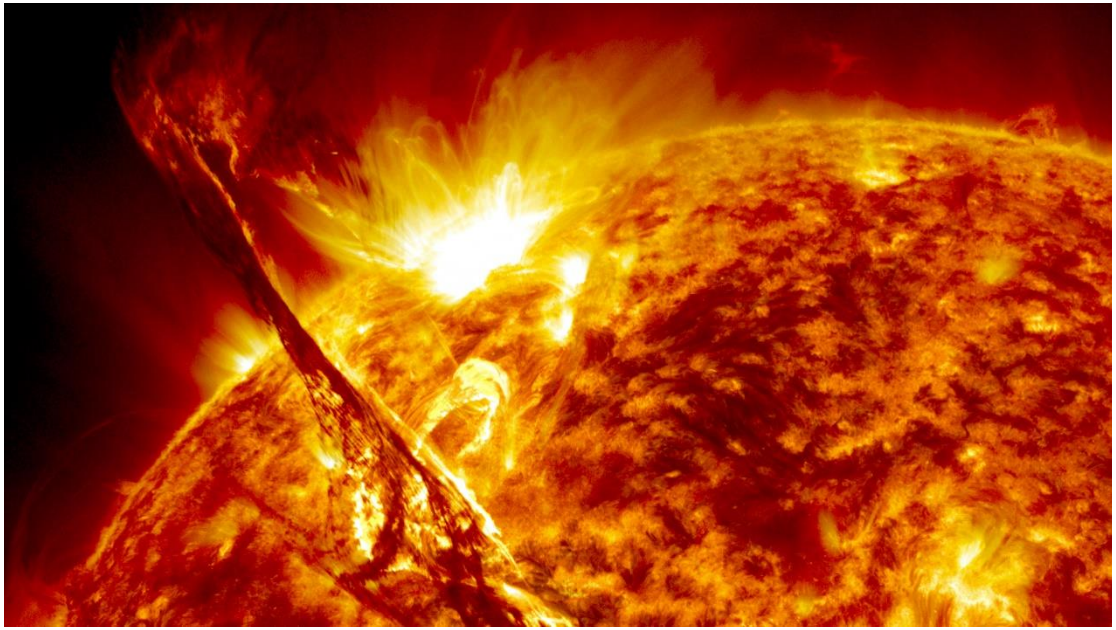
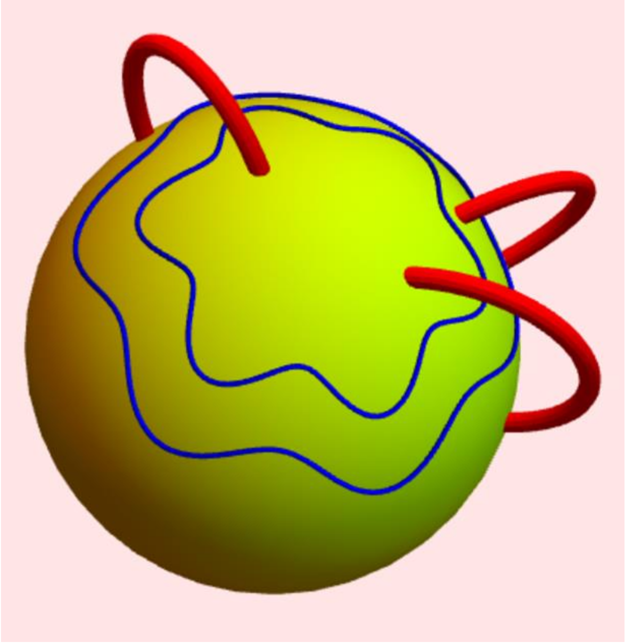
Absolute Helicity measures



- Generalizing the Poloidal-Toroidal Field Decomposition
- Helicity Flux through boundaries





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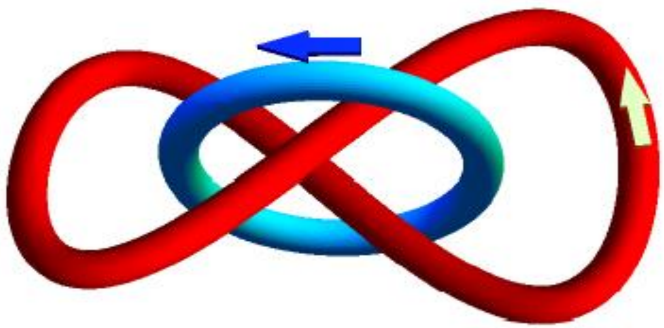
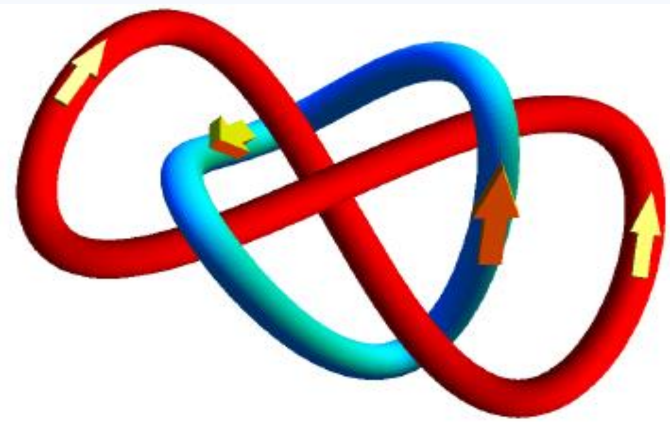
Gauss Linking Integral

$$L_{12} = -\frac{1}{4\pi} \oint_1 \oint_2 \frac{d\mathbf{x}}{d\sigma} \cdot \frac{\mathbf{r}}{r^3} \times \frac{d\mathbf{y}}{d\tau} \, d\tau \; d\sigma \qquad (\mathbf{r} = \mathbf{x} - \mathbf{y})$$

Magnetic Helicity Integral 1961 Moreau, 1969 Moffatt

$$H = -\frac{1}{4\pi} \int \int \mathrm{B}(\mathrm{x}) \cdot \frac{\mathbf{r}}{r^3} \times \mathrm{B}(\mathrm{y}) \; d^3y \; \mathrm{d}^3x$$





The Biot Savart integral (Coulomb gauge) 2001 Cantarella, DeTurck & Gluck

$$\mathrm{A}(\mathrm{x}) = -\frac{1}{4\pi} \int \frac{\mathbf{r}}{r^3} \times \mathrm{B}(\mathrm{y}) \; d^3y$$

$$\Rightarrow H = \int \mathrm{A} \cdot \mathrm{B} \; \mathrm{d}^3x.$$

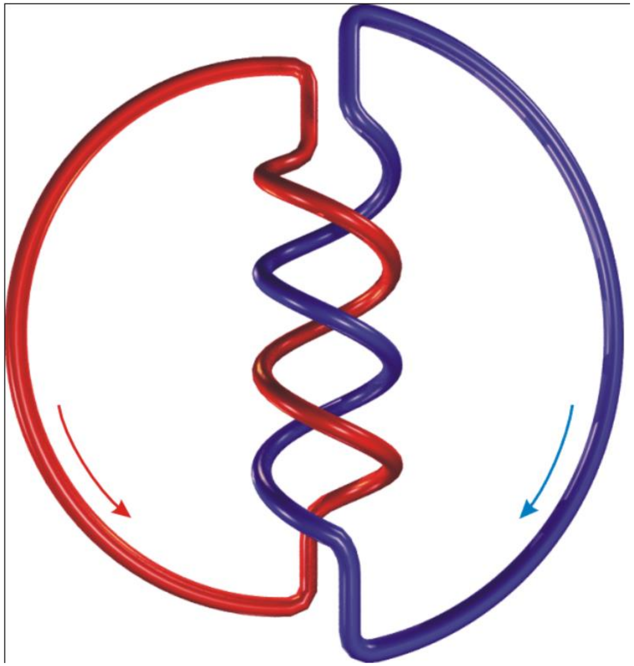
So is helicity a six-dimensional integral, or a three-dimensional integral?

Answer: neither - it really is five dimensional!

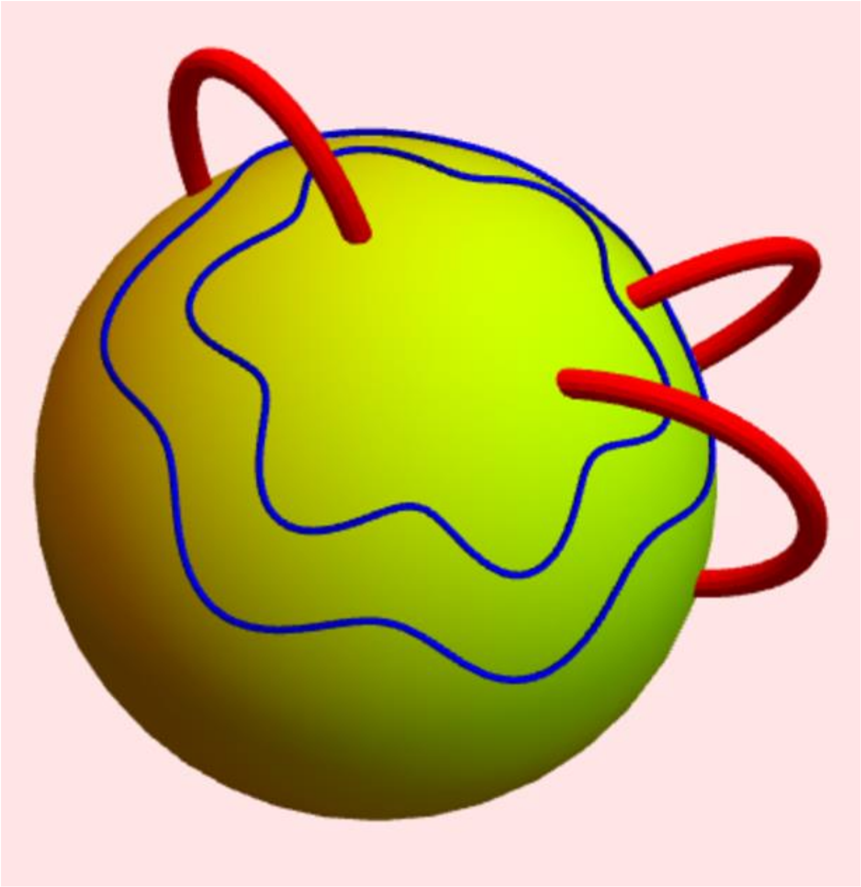
1. The linking number looks like a double integral, but we can do this as a single integral over mutual winding.

$$L_{12} = -\frac{1}{4\pi}\oint_1 \oint_2 \frac{d\mathbf{x}}{d\sigma} \cdot \frac{\mathbf{r}}{r^3} \times \frac{d\mathbf{y}}{d\tau}\, d\tau\ d\sigma \qquad (\mathbf{r} = \mathbf{x} - \mathbf{y})$$

$$L_{12} = \sum_{1_i} \sum_{2_j} \int \frac{\mathrm{d}\phi_{ij}}{\mathrm{d}z}\, \mathrm{d}z$$



2. Or, we can express the helicity as linking between poloidal and toroidal fields



Linking of Poloidal flux (red) by Toroidal flux (blue)

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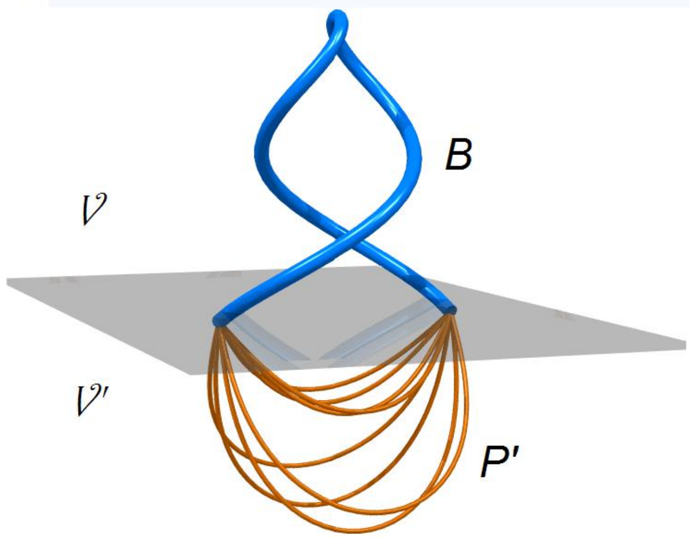
Helicity in Open volumes

The helicity is measured relative to that of the potential field P, where

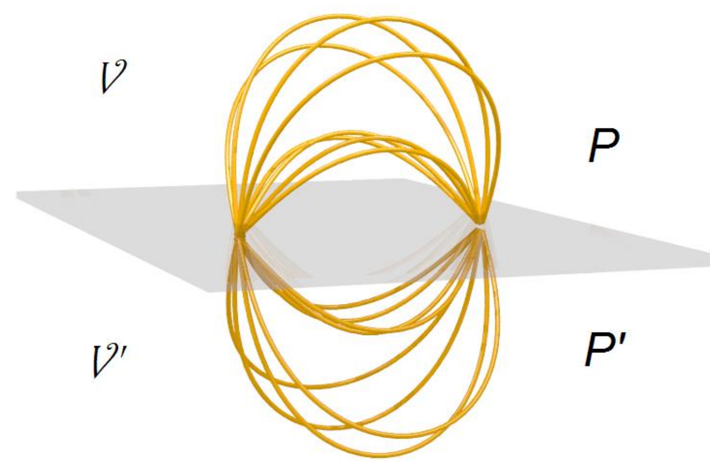
$$\nabla \times P = 0, \qquad P \cdot \hat{n} = B \cdot \hat{n}.$$

The result is independent of the field below the photosphere.

B & Field 84.



True Field



Reference Field

Equivalently, we set the helicity of the potential field to zero (and assume helicity is bilinear).

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Absolute Measures of Magnetic Helicity

Several Authors have explored alternative measures of helicity in open volumes (e.g. Hornig, Prior & Yeates, Low).

- Alternative Reference Fields
- Unique Vector Potentials
- Field Representations

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Alternative Reference Fields

Consider a single coronal flux tube stretching between two parallel planes occupying a volume $\mathcal{V}$. We have a choice of *two* potential fields as reference:

1) the potential field inside $\mathcal{V}$
2) the potential field between the two planes.

The first case is interesting for piece-wise constant-alpha force free fields.

The second case gives the sum of mutual winding numbers between field lines – it is equivalent to employing the winding gauge (Prior & Yeates 2014).

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Generalizing the Toroidal-Poloidal Decomposition to arbitrary simply connected surfaces

recall for spheres and planes:

$$\mathcal{L} \equiv \begin{cases} -\hat{z} \times \nabla, & \text{Cartesian geometry;} \\ -\hat{r} \times \nabla, & \text{Spherical geometry.} \end{cases} \tag{8}$$

Then one can show that functions T and P (the *toroidal* and *poloidal* flux functions) exist where

$$\mathbf{B}_T \equiv \mathcal{L}T \tag{9}$$

$$\mathbf{B}_P \equiv \nabla \times \mathcal{L}P. \tag{10}$$

The poloidal and toroidal fields are orthogonal in the sense that

$$\int \mathbf{B}_T \cdot \mathbf{B}_P \, \mathrm{d}^3x = 0.$$

Helicity as linking of Poloidal – Toroidal Fields

Physical Meaning: Poloidal and Toroidal Fields link each other, but not themselves. B85, Low 2010

Let

$$\begin{aligned} \mathcal{L} &\equiv \hat{z} \times \nabla \qquad \text{planar} \\ &\equiv \hat{r} \times \nabla \qquad \text{spherical} \end{aligned}$$

and

$$\mathbf{B} = \mathcal{L}T + \nabla \times \mathcal{L}P$$

The helicity between planes $z = z_1$ and $z = z_2$ can be written

$$\boxed{H = 2 \int \mathcal{L}T \cdot \mathcal{L}P \; \mathrm{d}^3x}.$$

We can write this as

$$\boxed{H = \int_{z_1}^{z_2} F(z) \; \mathrm{d}z},$$

$$F(z) = 2 \int \mathcal{L}T \cdot \mathcal{L}P \;\; \mathrm{d}x \;\; \mathrm{d}y.$$

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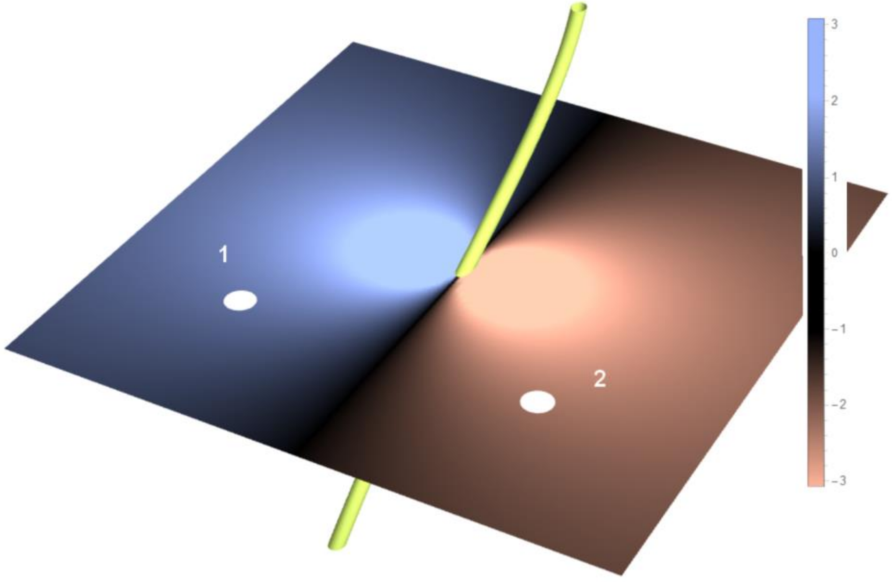
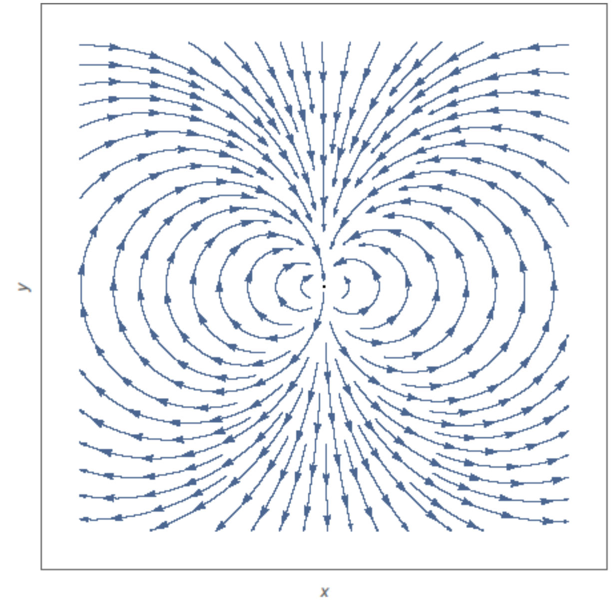
The toroidal potential and field due to a single oblique flux tube





The normal component of the curl

The fields $\mathbf{B}_P$ and $\mathbf{B}_T$ are determined by the boundary data B_n or J_n. We can express this idea by defining the normal component of the curl as the operator

$$\mathcal{D}\mathbf{V} = \hat{\mathbf{n}} \cdot \nabla \times \mathbf{V}. \tag{4}$$

Next consider its inverse operator $\mathcal{D}^{-1}$: given a scalar function $f(x, y)$ or $f(\theta, \phi)$ on a planar or spherical surface, we specify that the inverse normal curl gives a divergence-free vector field parallel to the surface: if $\mathbf{V} = \mathcal{D}^{-1} f$ then

$$\nabla \cdot \mathbf{V} = 0; \qquad \hat{\mathbf{n}} \cdot \mathbf{V} = 0. \tag{5}$$

This field is unique: if two fields $\mathbf{V}_1$ and $\mathbf{V}_2$ both satisfy these equations then $\mathbf{V}_2 - \mathbf{V}_1$ would be a gradient field, $(\mathbf{V}_2 - \mathbf{V}_1) = \nabla_{\|}\psi$ with zero divergence. Thus the two-dimensional Laplacian $\Delta_{\|}\psi = 0$ where the only solutions are $\psi =$ constant.

Hence we can write

$$\mathbf{B}_T = \mathcal{D}^{-1} J_n \tag{6}$$

$$\mathbf{B}_P = \nabla \times \mathcal{D}^{-1} B_n. \tag{7}$$

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Arbitrary Simply Connected Volumes

Let *w be a 'radial' coordinate* labelling nested surfaces. The poloidal field will contain all the normal flux. The toroidal field will contain all the normal current.

The Toroidal Field

On each w surface we define the toroidal field to be the inverse curl of the normal current:

$$\mathbf{B}_T = \mathcal{D}^{-1} J_n. \tag{32}$$

The Poloidal Field

We can define the poloidal field as whatever else remains after subtracting the toroidal field:

$$\mathbf{B}_P \equiv \mathbf{B} - \mathbf{B}_T. \tag{34}$$

This ensures that $B_{Pn} = B_n$ and $J_{Pn} = 0$.

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In non-symmetric volumes the poloidal field doesn’t behave properly!

In particular, the obvious vector potential

$$\widetilde{\mathbf{A}} = \mathcal{D}^{-1} B_n$$

gives an unwanted field parallel to the surface, and hence a spurious extra perpendicular current. We need to add a correction term – a new toroidal field:

$$\mathbf{B}_P = \widetilde{\mathbf{B}} - \mathbf{B}_\mathcal{S} \tag{36}$$

where $\mathbf{B}_\mathcal{S}$ is a toroidal field with the same undesired perpendicular current:

$$\mathcal{D}\mathbf{B}_\mathcal{S} = \mathcal{D}\widetilde{\mathbf{B}} = \mathcal{D}\nabla \times \mathcal{D}^{-1} B_n. \tag{37}$$

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The shape field

The extra toroidal field arising in the poloidal decomposition can be called the *shape field.* It is only non-zero if the curvature is not constant.

Conclusion: The helicity can still be defined as the linking of poloidal with toroidal fields.

But: there is now an extra linking, of the poloidal field with the shape field.

$$H_{\mathcal{V}} \equiv 2\int \widetilde{\mathbf{A}} \cdot \mathbf{B}_T \, \mathrm{d}^3x + \int \widetilde{\mathbf{A}} \cdot \mathbf{B}_{\mathcal{S}} \, \mathrm{d}^3x.$$

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The Helicity Flux

$$\frac{\mathrm{d}H_{\mathcal{V}}}{\mathrm{d}t} = -2\int_{\mathcal{V}} \mathbf{E}\cdot\mathbf{B}\ \mathrm{d}^3x + 2\oint_{S} \widetilde{\mathbf{A}}\times\mathbf{E}\cdot\hat{\mathbf{n}}\ d^2x$$

Here $\widetilde{\mathbf{A}}$ is the unique vector potential parallel to the boundary and divergence free.

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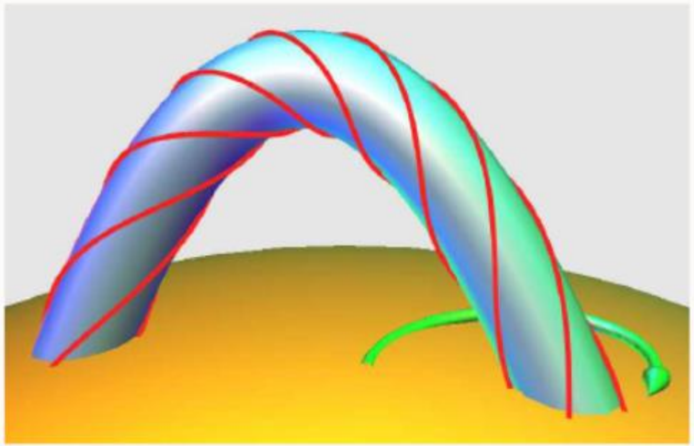
Helicity Flux

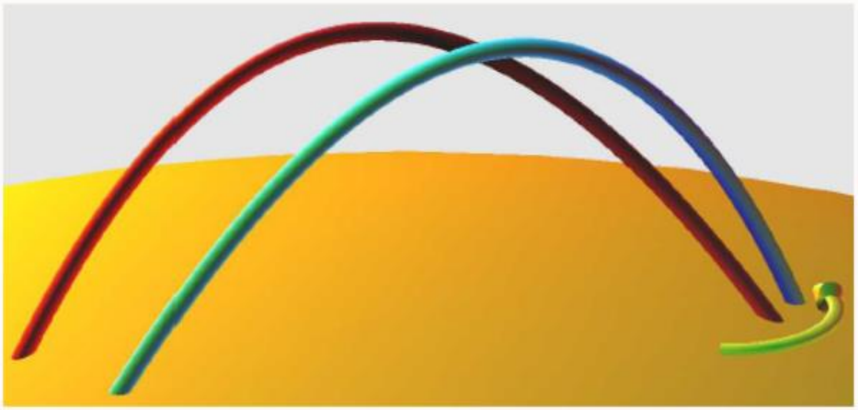
Consider a set of magnetic loops with endpoints on the solar surface. The helicity will change when the endpoints move parallel to the surface:

1. If the endpoints spin, that will twist the magnetic flux above.
2. If the endpoints orbit each other, that will add to mutual winding.

The orbit term involves the velocities at each end $\mathbf{V}_1$ and $\mathbf{V}_2$, the flux Φ, and the vector potentials generated by the two fluxes (Φ and $-\Phi$):

$$\frac{dH}{dt} = 2\Phi(\mathbf{A}_1(\mathbf{x_2}) \cdot \mathbf{V}_2 - \mathbf{A}_2(\mathbf{x_1}) \cdot \mathbf{V}_1)$$





Vector Potentials in a Plane

The vector potential in a plane at point $\vec{\mathbf{x}}_2$ due to a flux element at point $\vec{\mathbf{x}}_1$, flux Φ_1 is (with $\vec{\mathbf{r}}_{12} = \vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2$)

$$\vec{\mathbf{A}}(\vec{\mathbf{x}}_2) = \frac{\Phi_1}{2\pi}\hat{z} \times \frac{\vec{\mathbf{r}}_{12}}{|r_{12}|^2}.$$

Avoiding monopoles

As there should be no monopoles, the flux Φ_1 must return through the plane – but there is plenty of area in a plane so the return magnetic field is infinitesimal!

Vector Potentials on a Sphere

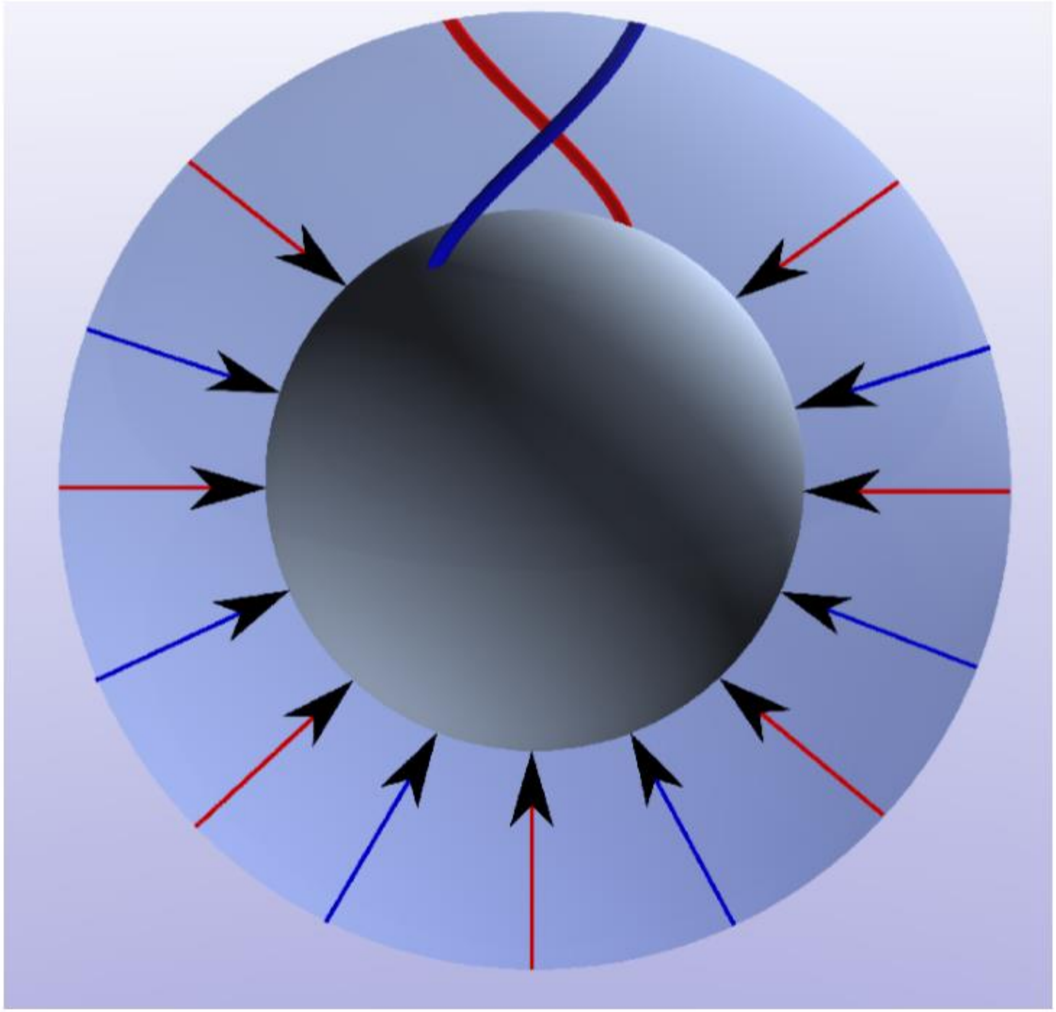


Figure 5. We introduce a net return flux to ensure that there are no magnetic monopoles. Each tube's B-field is shared out evenly and directed back into the surface.

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Vector Potentials on a sphere

$$\mathbf{B}_1(\theta,\phi) = \Phi_1 \begin{cases} \frac{1}{A_1} - \frac{1}{4\pi R^2} & \text{inside footpoint 1} \\ -\frac{1}{4\pi R^2} & \text{outside footpoint 1} \end{cases}$$

Employing Stoke's theorem, the vector potential is

$$\mathbf{A}_{P1}(\theta,\phi) = \left(\frac{1+\cos\theta}{2}\right)\frac{\Phi_1}{2\pi R\sin\theta}\hat{\phi}.$$

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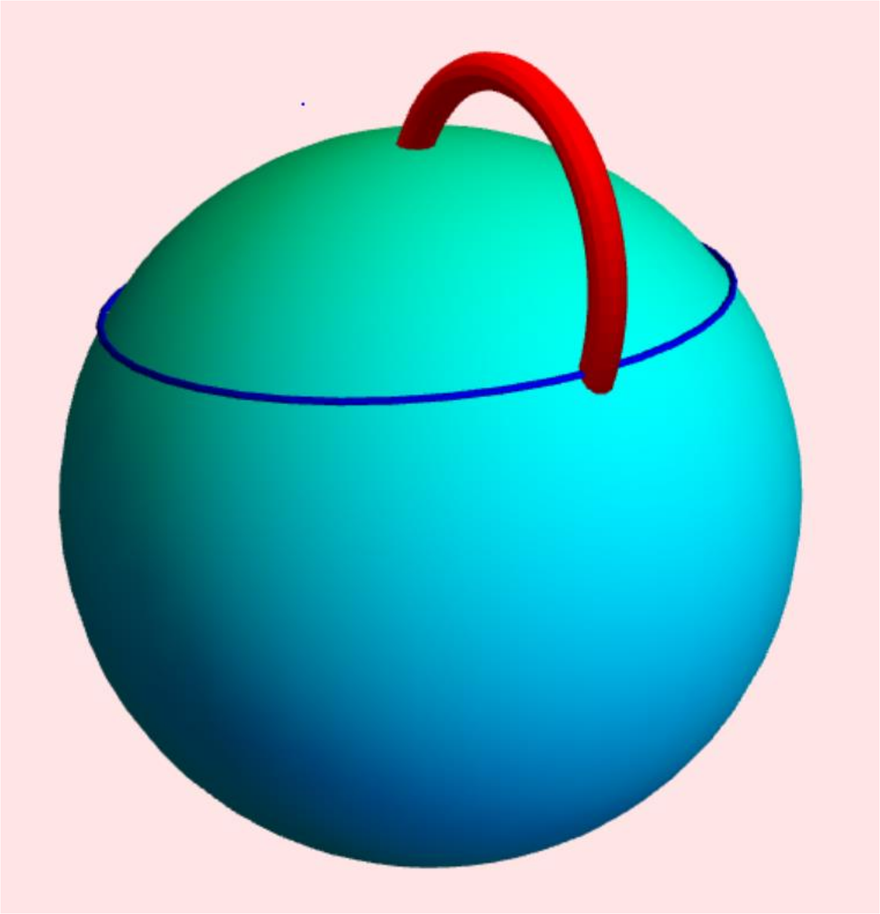

The helicity flux due to uniform rotation of the sphere should be zero.
Suppose the sphere goes through one full turn. The Orbit term gives

$$(1+\cos\theta)\,\Phi_1$$

Meanwhile, the footpoint at the pole spins through one turn, yielding a twist of -1.

The footpoint at co-latitude θ spins through $\delta\phi/2\pi$, *where*

$$\delta\phi = (2\pi \cos\ \theta)$$

We could rewrite this in terms of the *geodesic deviation*

$$\delta\phi = (2\pi - \text{geodesic deviation})$$

The geodesic deviation is simply $2\pi\,(1 - \cos\ \theta)$.

Gauss- Bonnet Theorem

The total curvature K enclosed by a closed curve on a simply connected surface is

$$K = 4\pi(1 - \text{geodesic deviation})$$

Conclusion: in any simply connected manifold, the return flux should be distributed according to Gauss curvature.

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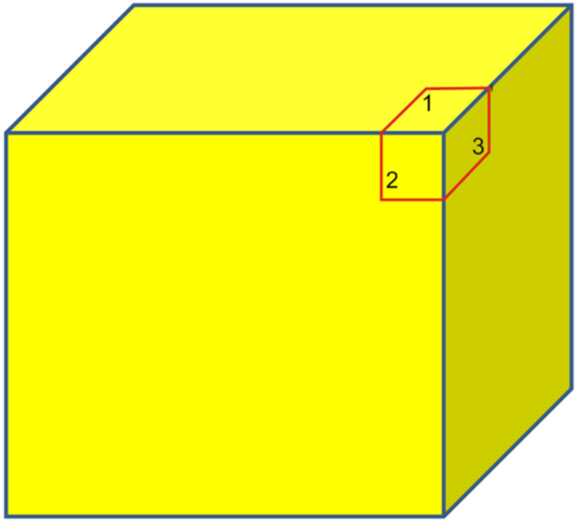
Surface vector potentials for footpoints

The net magnetic flux (or vorticity) through a compact boundary must be zero. Each footpoint needs compensating flux.

Distribute the return flux proportional to the Gauss curvature to correct geodesic deviation (angle deficits)!



Self and Mutual Helicity

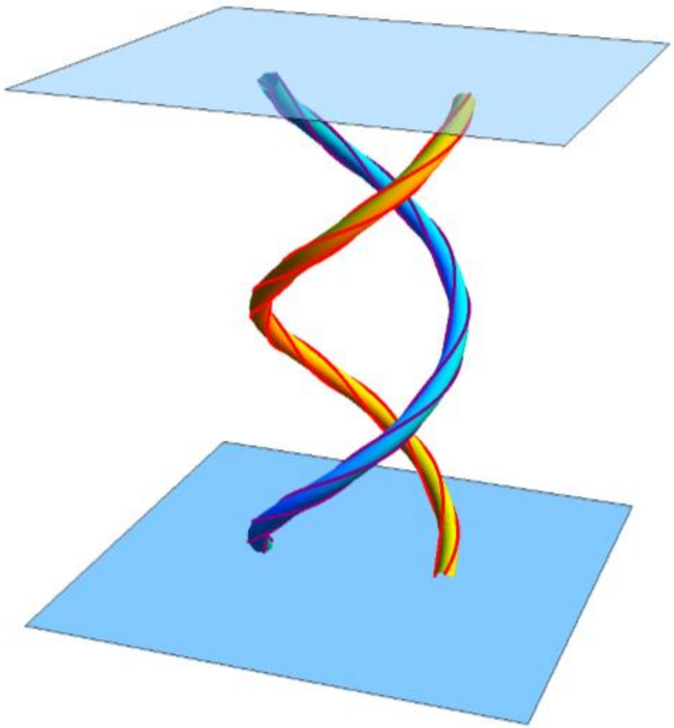


Figure 2. Two flux tubes extending between two parallel planes. The twist plus writhe of each individual tube is conserved, as well as the winding number between the two tubes.

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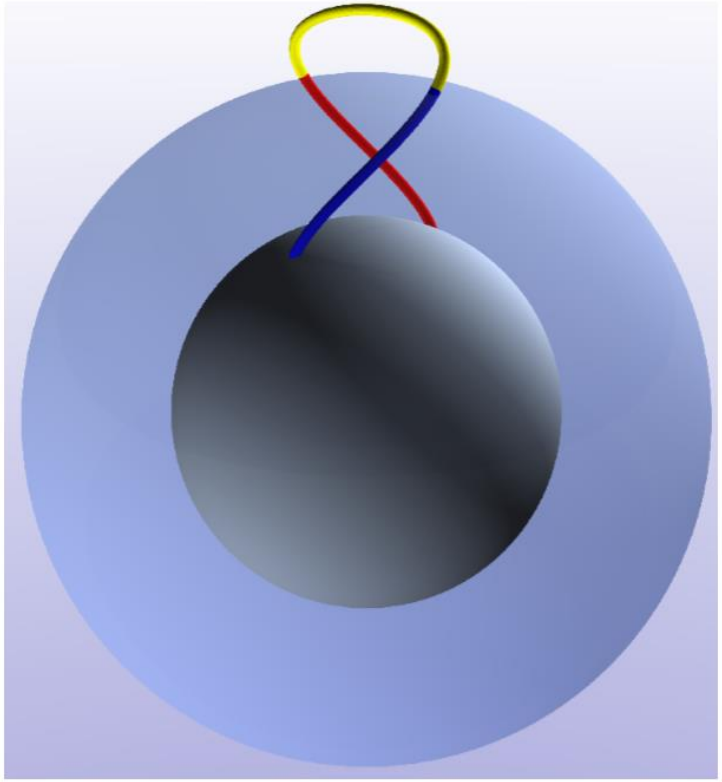
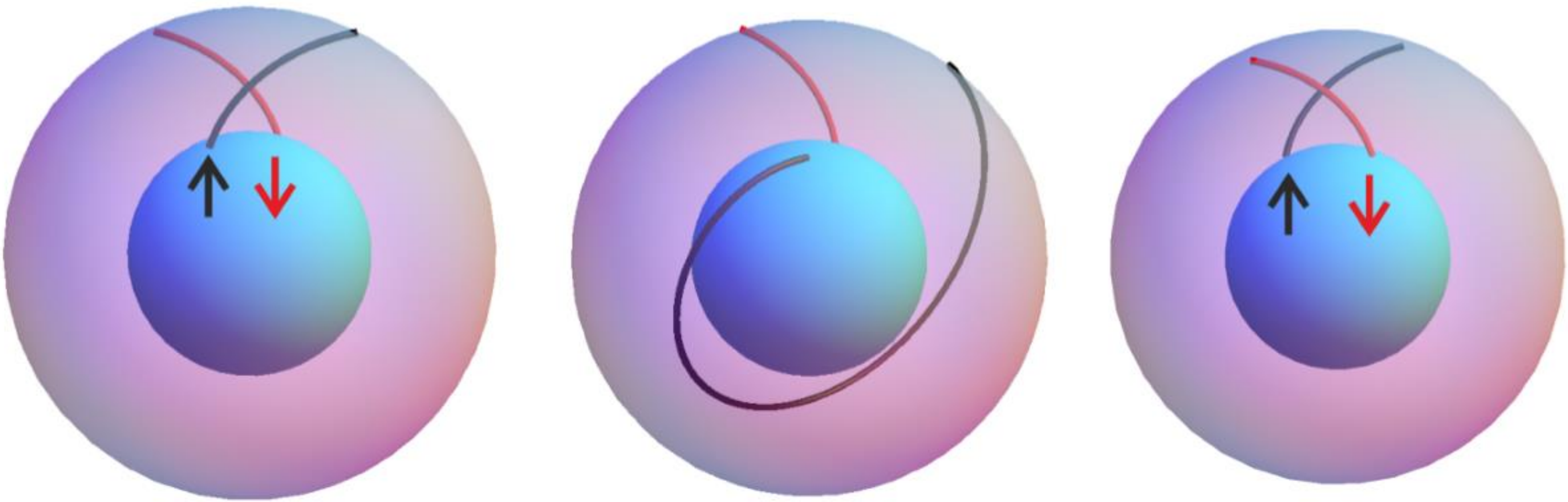
Helicity in Spherical Volumes

In outer spherical volumes, self helicity (twist+writhe) and mutual linking are not easy to separate!



Converting mutual to self helicity in a non-trivial topology



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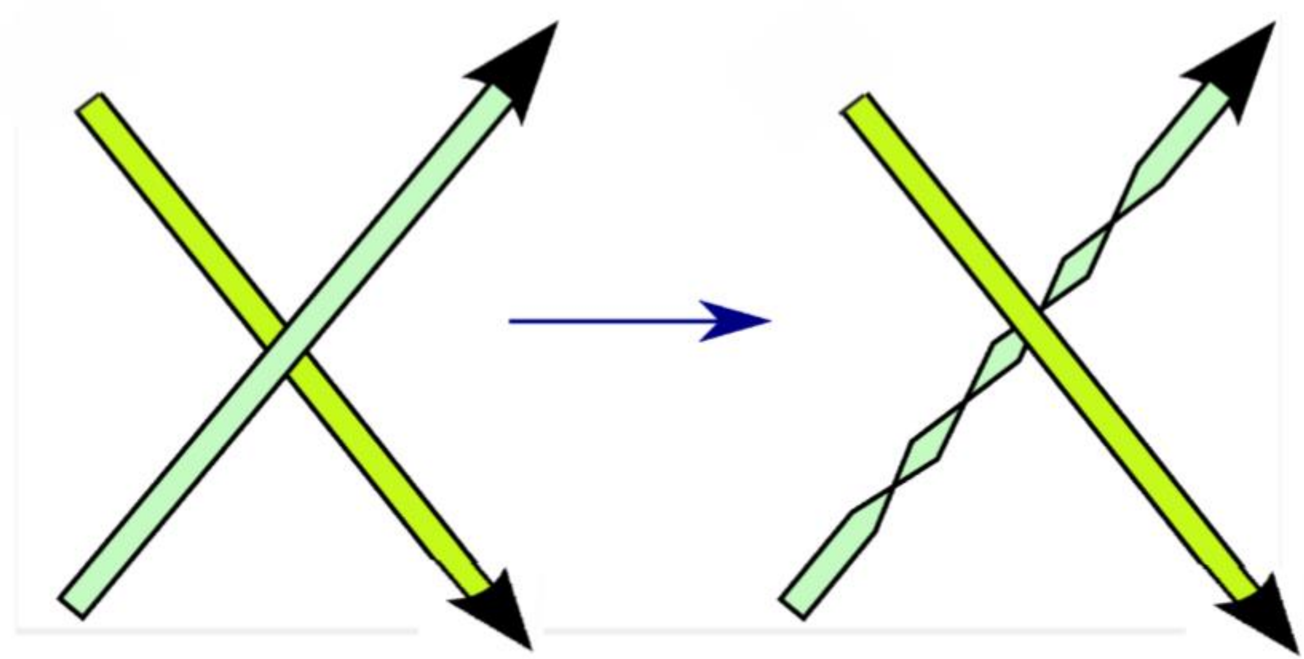

Figure 4. Before deforming: $\mathcal{T} = 0$ and $\mathcal{W} = -1$. Post deformation: $\mathcal{T} = -2$ and $\mathcal{W} = 1$. Hence in both cases $H = -\Phi^2$ as expected.

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