Topology Conserving Magnetic Field Evolution

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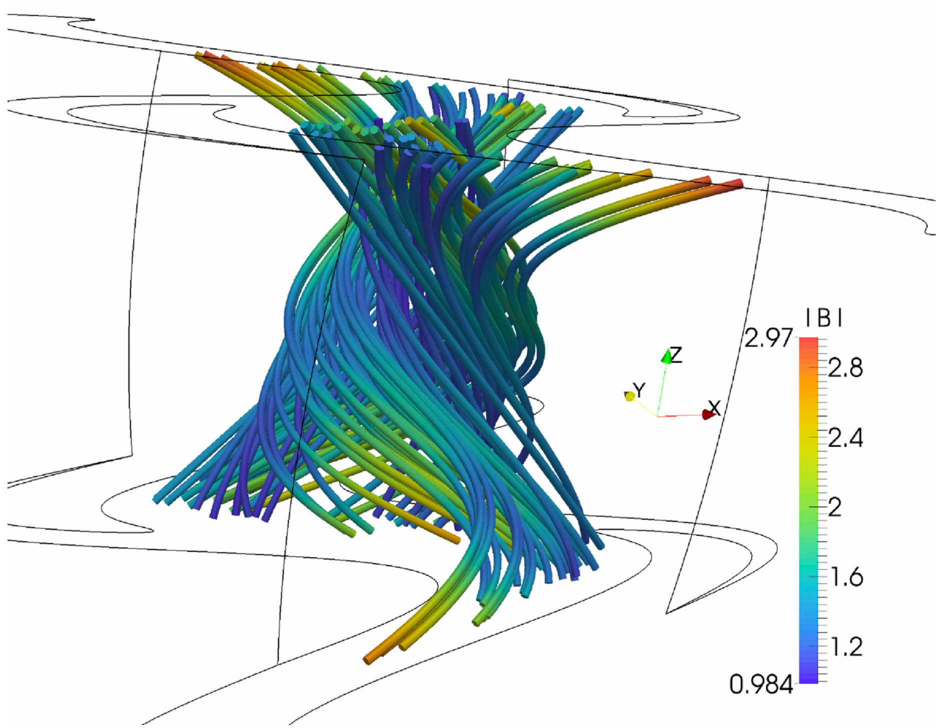
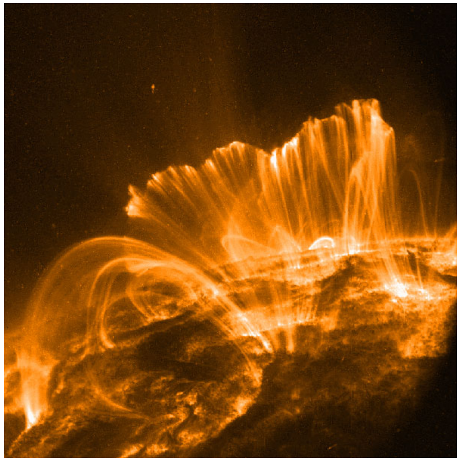


Magnetic Field Relaxation

Solar corona: low plasma beta and magnetic resistivity

NASA

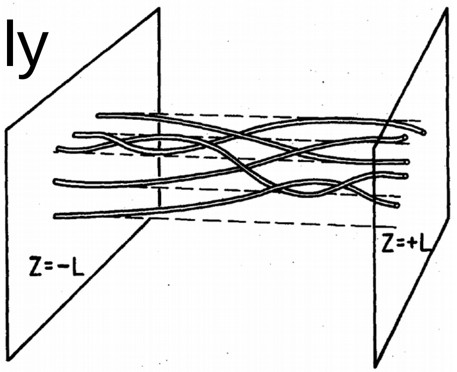


- Force-free magnetic fields
- Minimum energy state

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \;\Leftrightarrow\; \nabla \times \mathbf{B} = \alpha \mathbf{B}$$

Parker: Equilibrium with the same topology exists only if the twist varies uniformly along the field lines. Strongly braided fields → topological dissipation.

(Parker 1972)



Braided fields from foot point motion complex enough. *(Parker 1983)*

Solutions possible with filamentary current structures (sheets).

(Mikic 1989, Low 2010)

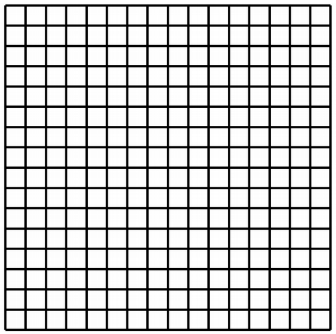
Ideal Field Relaxation

Ideal induction eq.: $$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{0}$$

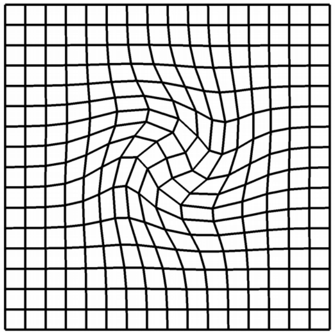
Lie-transport:

$$\frac{\partial}{\partial t}\beta(\mathbf{X},t) + \mathcal{L}_{\mathbf{u}}\beta(\mathbf{X},t) = 0$$

$$\mathbf{x}(\mathbf{X},0) = \mathbf{X}$$



$$\mathbf{x}(\mathbf{X},t)$$



Pull-back:

$$(\mathbf{x}^*(t)\beta)(\mathbf{X},t) = \beta(\mathbf{X},t)$$

Field evolution: $$B_i(\mathbf{X},t) = \frac{1}{\Delta}\sum_{j=1}^{3}\frac{\partial x_i}{\partial X_j}B_j(\mathbf{X},0)$$

$$\Delta = \det\left(\frac{\partial x_i}{\partial X_j}\right)$$

Preserves topology and divergence-freeness.

Ideal Field Relaxation

Magneto-frictional term: $\mathbf{u} = \mathbf{J} \times \mathbf{B}$ $\mathbf{J} = \nabla \times \mathbf{B}$

$$\Rightarrow \frac{\mathrm{d}E_{\mathrm{M}}}{\mathrm{d}t} < 0$$ *(Craig and Sneyd 1986)*

Fluid with pressure: $\mathbf{u} = \mathbf{J} \times \mathbf{B} - \beta \nabla \rho$

Fluid with inertia: $\mathrm{d}\mathbf{u}/\mathrm{d}t = (\mathbf{J} \times \mathbf{B} - \nu \mathbf{u} - \beta \nabla \rho)/\rho$

For $\mathbf{J} = \nabla \times \mathbf{B}$ use mimetic numerical operators.

(Hyman, Shashkov 1997)

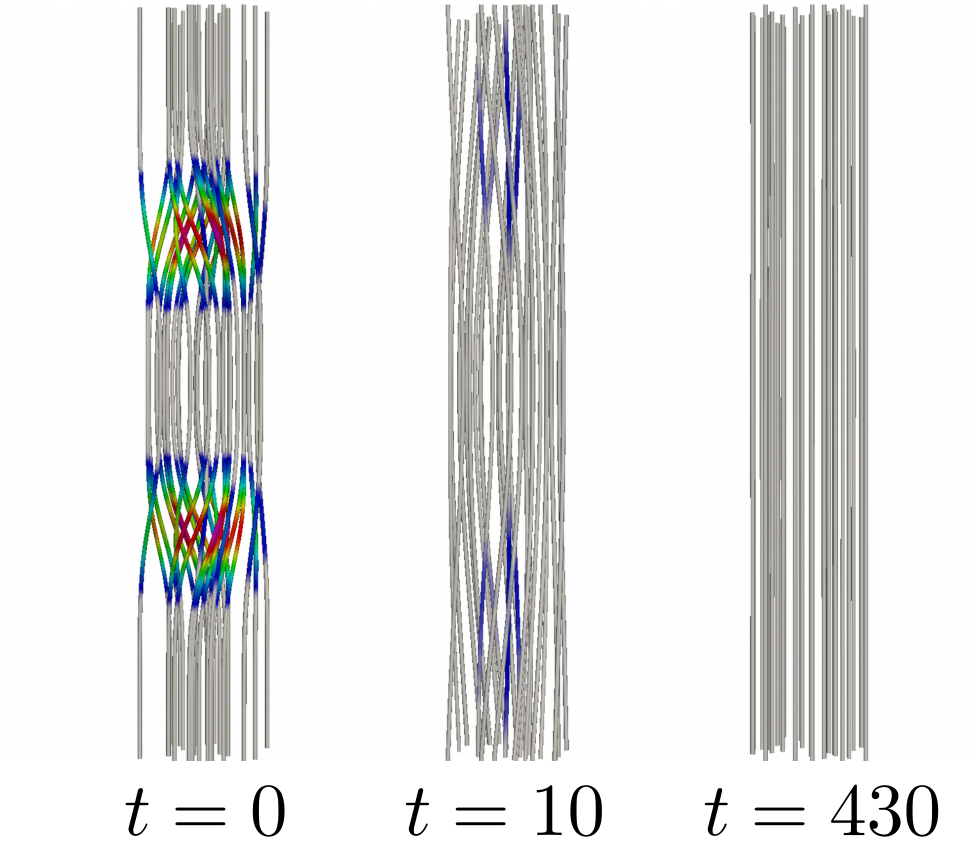
$$\nabla_{\mathrm{m}} \cdot \nabla_{\mathrm{m}} \times \mathbf{B} = 0 \qquad \nabla_{\mathrm{m}} \times \nabla_{\mathrm{m}} f = 0$$

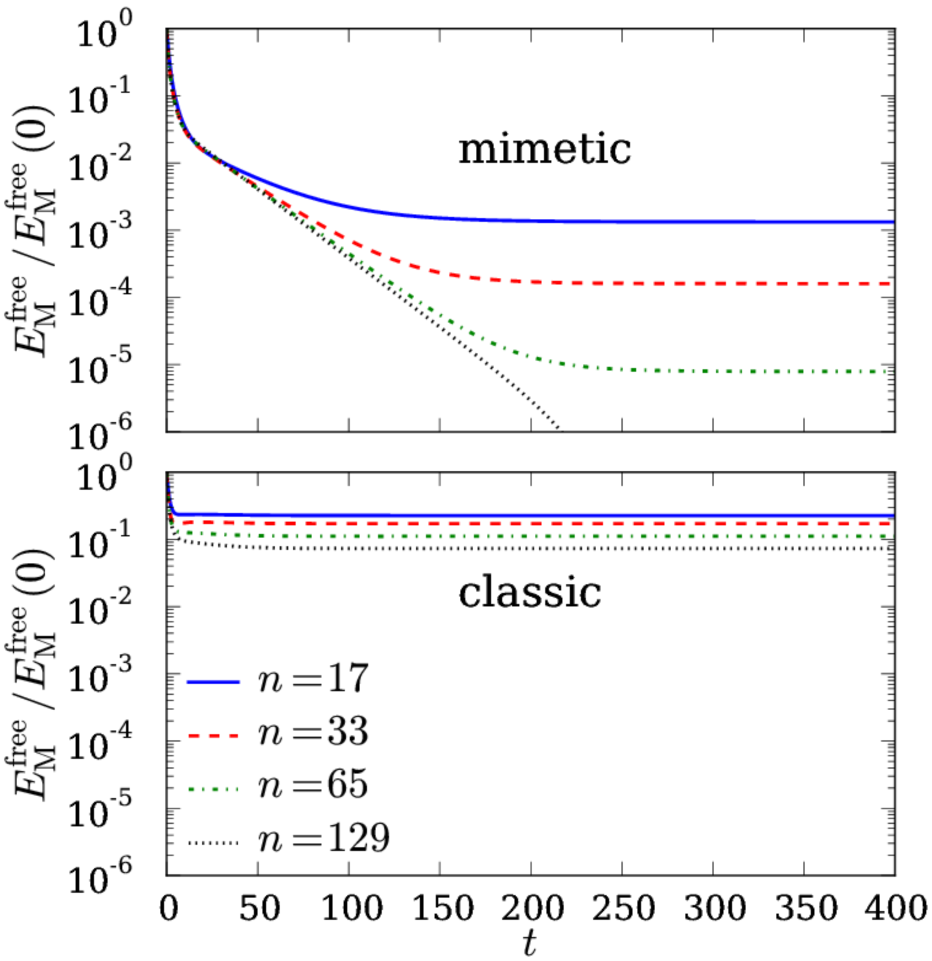
Own GPU code GLEMuR: (https://github.com/SimonCan/glemur)

(Candelaresi et al. 2014)

Proof of Concept

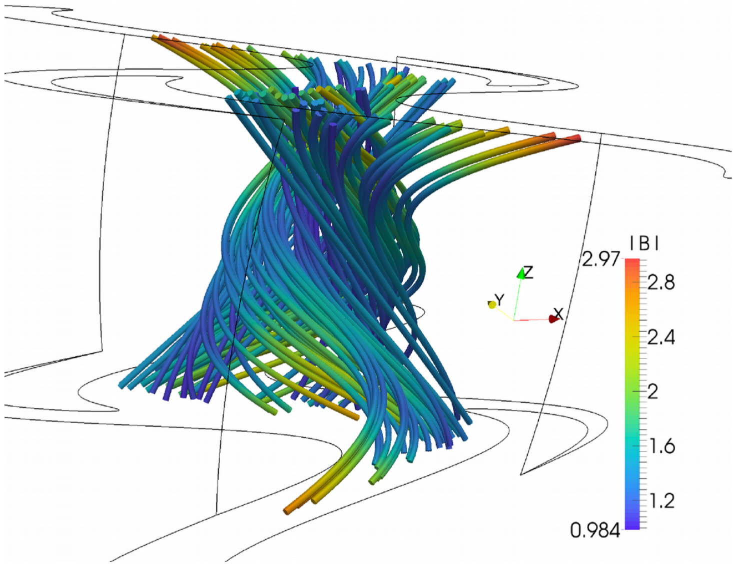
Magnetic streamlines:

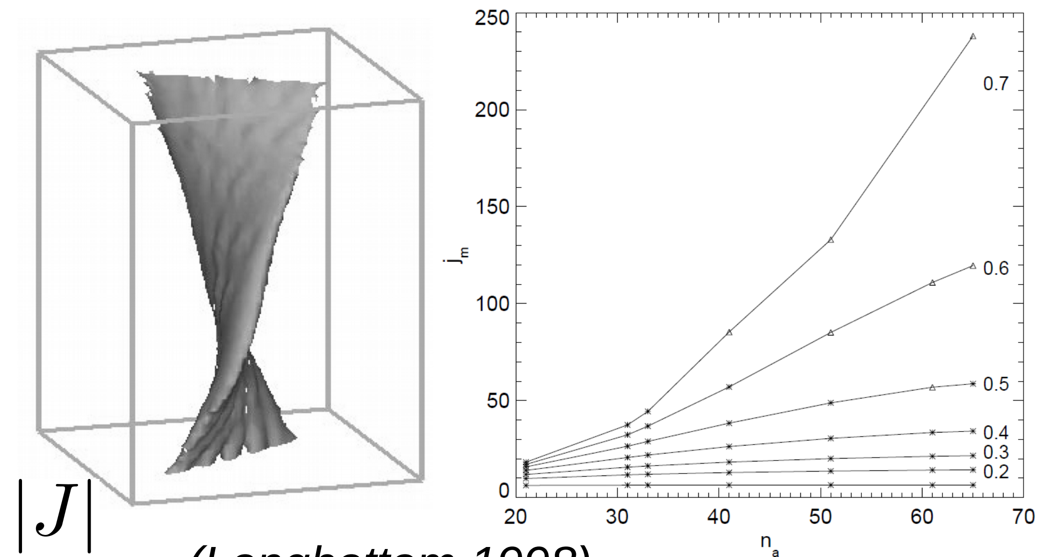




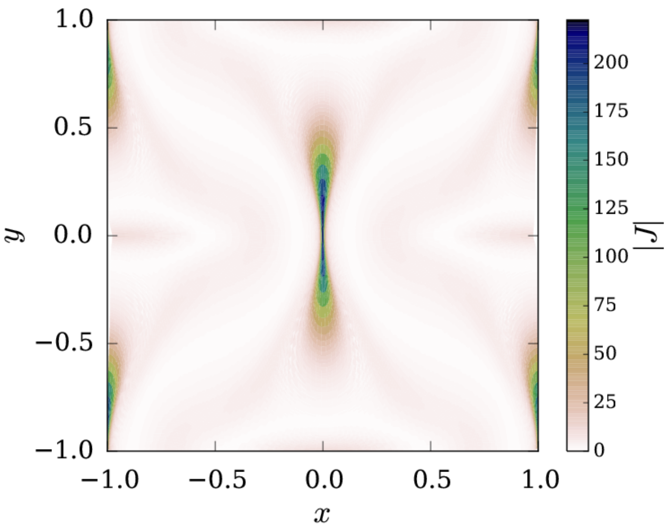
(Candelaresi et al. 2014)

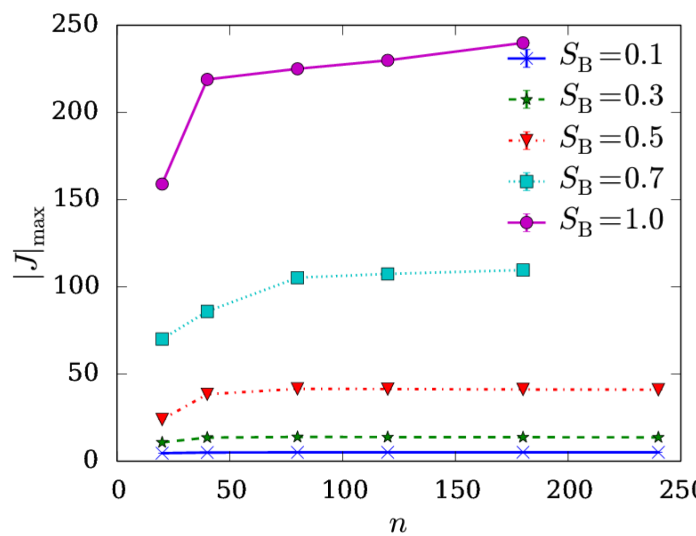
Distorted Magnetic Fields





(Longbottom 1998)

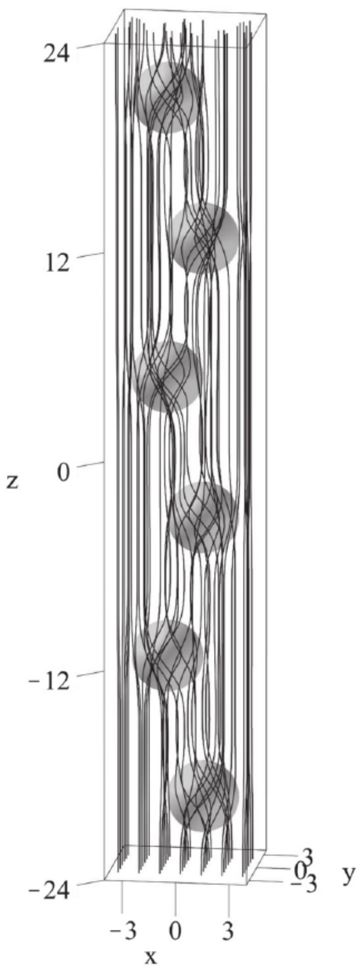


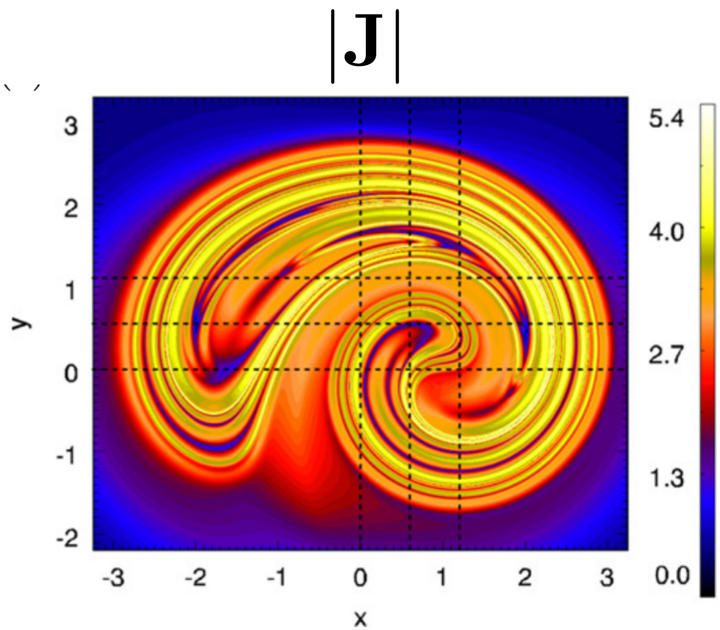


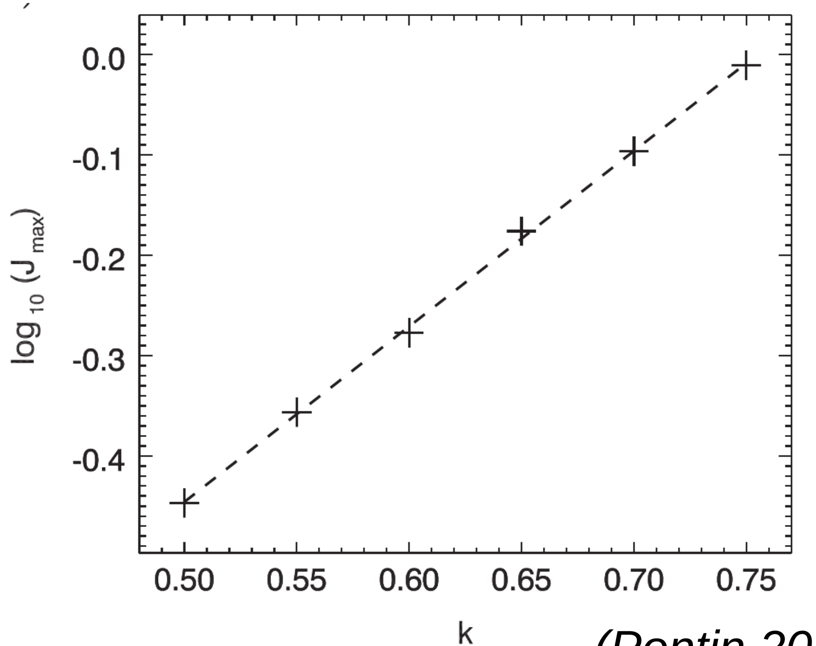
- resolved current concentrations
- shear leads to strong currents

(Candelaresi et al. 2015)

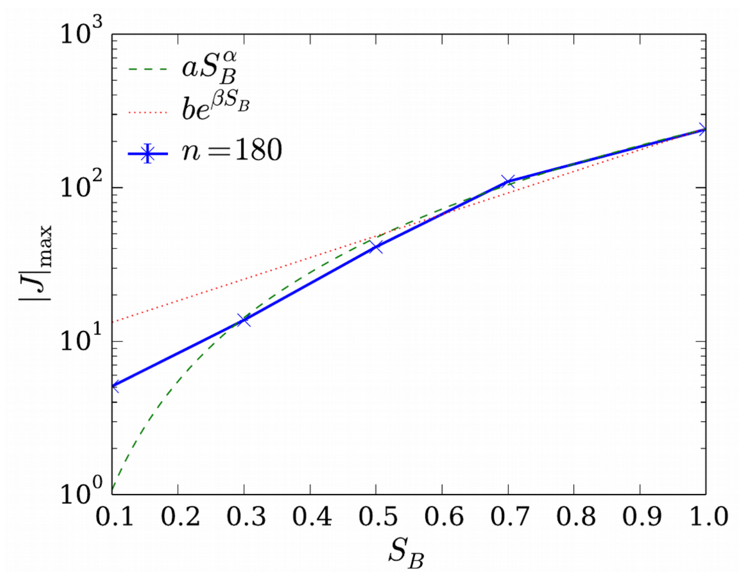
Exponential Increase in Current







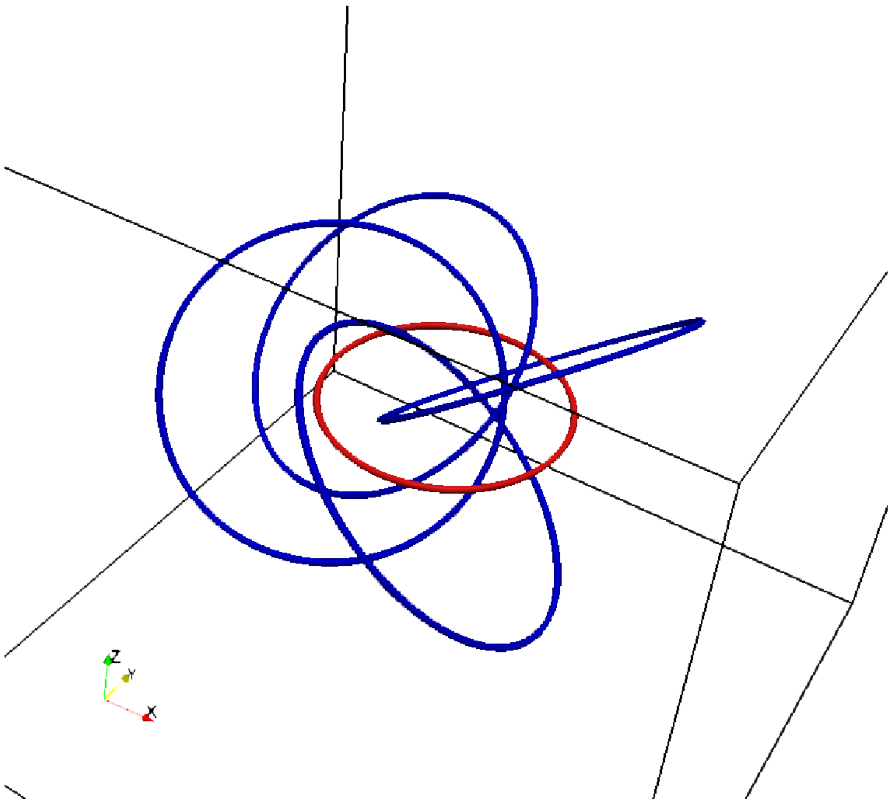
(Pontin 2015)



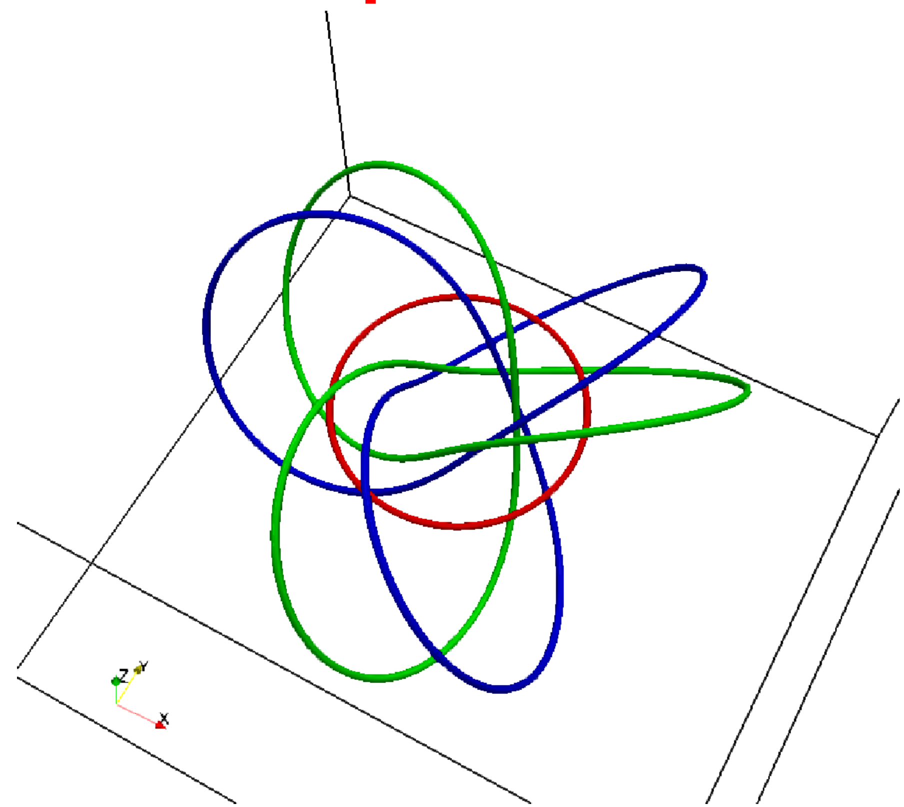
(Candelaresi 2015)

- lengths decrease exp. with complexity
- current increases exp. with complexity

Ideal Relaxation of the Hopf Fibration



$(\omega_1, \omega_2) = (1, 1)$



$(\omega_1, \omega_2) = (3, 2)$

$$\mathbf{B}_{\omega_1,\omega_2} = \frac{4\sqrt{s}}{\pi(1+r^2)^3\sqrt{\omega_1^2+\omega_2^2}} \begin{pmatrix} 2(\omega_2 y - \omega_1 x z) \\ -2(\omega_2 x + \omega_1 y z) \\ \omega_1(-1 + x^2 + y^2 - z^2) \end{pmatrix}$$

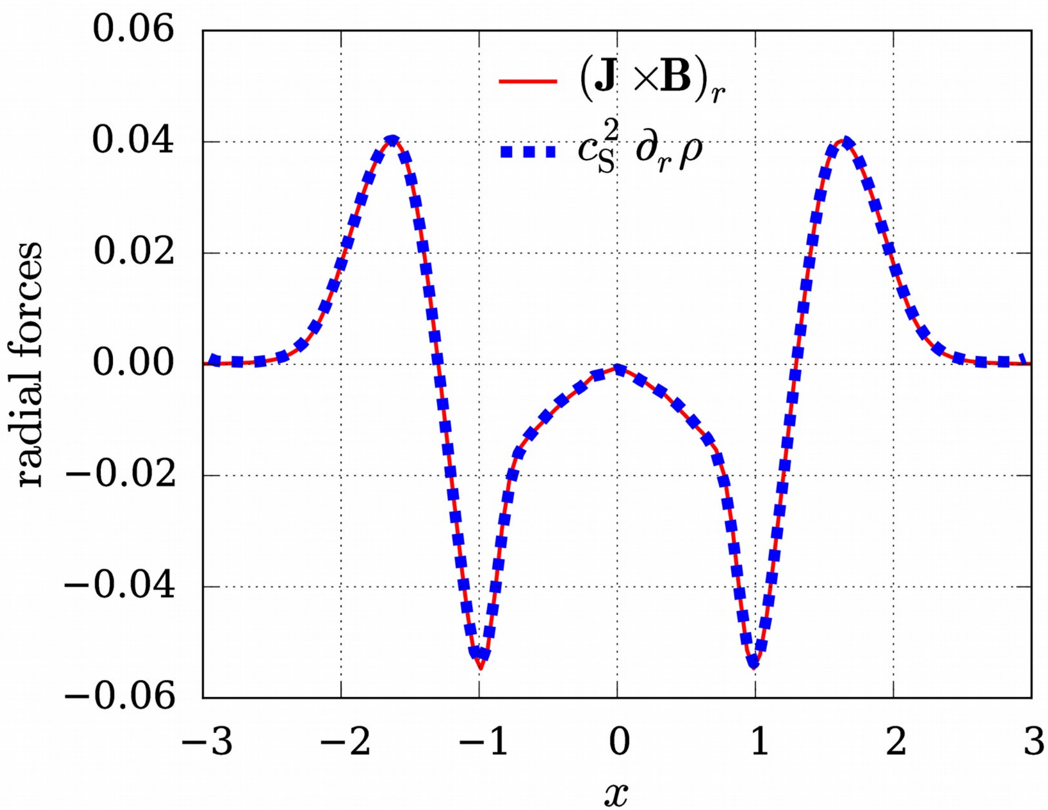
Ideal Relaxation of the Hopf Fibration

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{0}$$

Ideal MHD.

$$\mathrm{d}\mathbf{u}/\mathrm{d}t = (\mathbf{J} \times \mathbf{B} - \nu \mathbf{u} - \beta \nabla \rho)/\rho$$

Momentum equation with velocity damping.



- Force balanced field (Berg-Smiet et al. 2017).
- Grad-Shafranov solutions (Shafranov, 1966).

Conclusions

- Topology preserving relaxation of magnetic fields.
- GLEMuR: (https://github.com/SimonCan/glemur)
- Current concentrations not singular.
- Current increases strongly with field complexity.
- Hopf fibrations relax into Grad-Shafranov states in ideal MHD.

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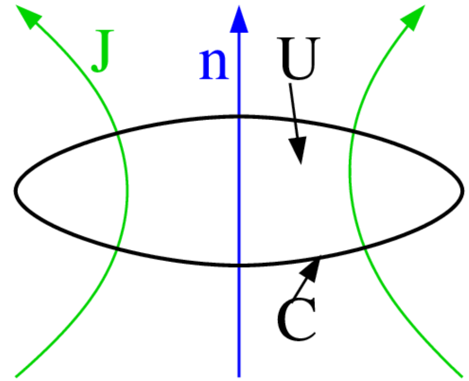
Mimetic Numerical Operators

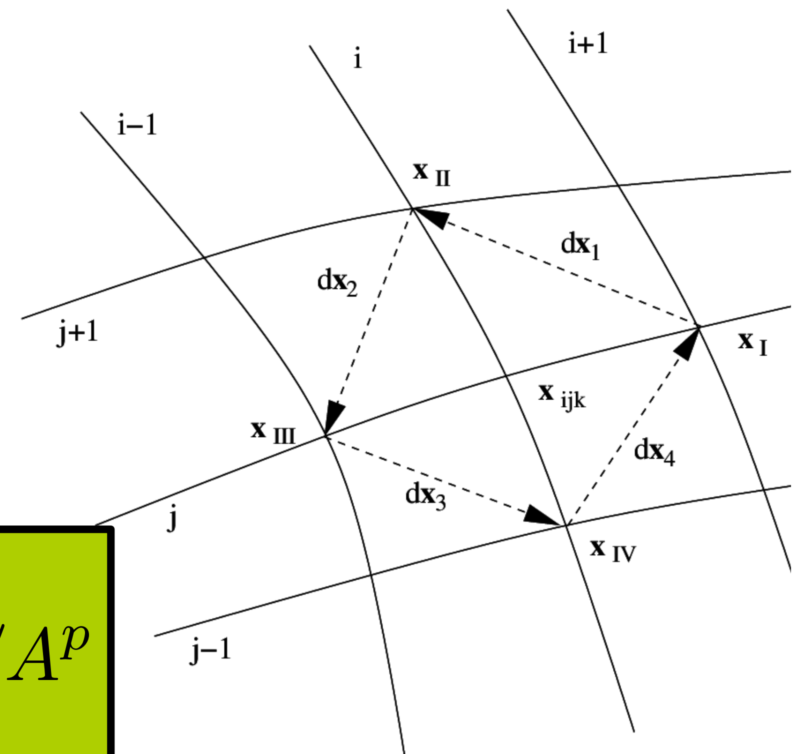
$$I = \int_U \mathbf{J}\cdot\mathbf{n}\,\mathrm{d}S = \oint_C \mathbf{B}\cdot\mathrm{d}\mathbf{r}$$



Discretized:

$$I \approx \mathbf{J}(\mathbf{X}_{ijk})\cdot\mathbf{n}A = \sum_{r=1}^{4}\mathbf{B}_r\cdot\mathrm{d}\mathbf{x}_r$$

$$\mathbf{J}(\mathbf{X}_U) \approx \mathbf{J}(\mathbf{X_{ijk}}), \quad \mathbf{X}_U \in U$$



3 planes will give 3 l.i. normal vectors:

$$I^p = \mathbf{J}(\mathbf{X}_{ijk})\cdot\mathbf{n}^p = \sum_{r=1}^{4}\mathbf{B}_r^p\cdot\mathrm{d}\mathbf{x}_r/A^p$$

Inversion yields $\mathbf{J}$ with $\nabla\cdot\mathbf{J} = 0$.

(Hyman, Shashkov 1997)

Quality Parameters

For a force-free field: $\nabla \times \mathbf{B} = \alpha \mathbf{B}$

$$\mathbf{B} \cdot \nabla \alpha = 0$$

- Force-free parameter does not change along field lines.
- Measure the change of $\alpha^* = \dfrac{\mathbf{J} \cdot \mathbf{B}}{\mathbf{B}^2}$ along field lines:

$$\epsilon^* = \max_{i,j} \left(a_r \frac{\alpha^*(\mathbf{X}_i) - \alpha^*(\mathbf{X}_j)}{|\mathbf{X}_i - \mathbf{X}_j|} \right); \quad \mathbf{X}_i, \mathbf{X}_j \in s_\alpha$$

Particular field line: $s_\alpha = \{(0, 0, Z) : Z \in [-L_z/2, L_z/2]\}$

Quality Parameters

Deviation from the expected relaxed state:

$$\sigma_{\mathbf{x}} = \sqrt{\frac{1}{N} \sum_{ijk} (\mathbf{x}(\mathbf{X}_{ijk}) - \mathbf{x}_{\mathrm{relax}}(\mathbf{X}_{ijk}))^2}$$

$$\sigma_{\mathbf{B}} = \sqrt{\frac{1}{N} \sum_{ijk} (\mathbf{B}(\mathbf{X}_{ijk}) - \mathbf{B}_{\mathrm{relax}}(\mathbf{X}_{ijk}))^2}$$

Free magnetic energy:

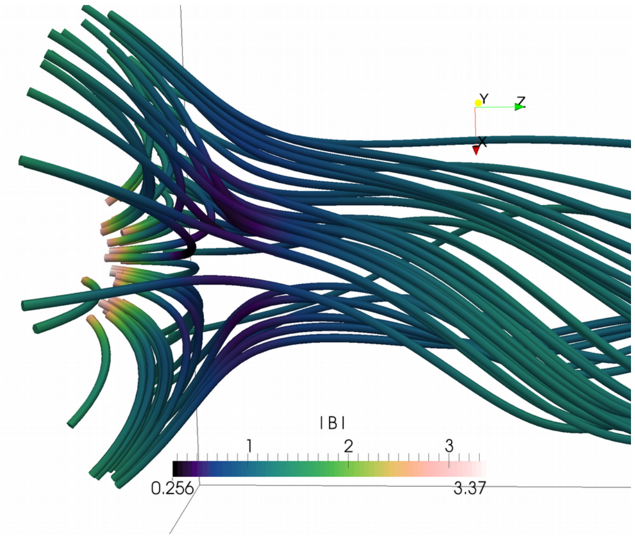
$$E_{\mathrm{M}}^{\mathrm{free}} = E_{\mathrm{M}} - E_{\mathrm{M}}^{\mathrm{bkg}}$$

$$E_{\mathrm{M}} = \int_V \mathbf{B}^2/2 \ \mathrm{d}V \qquad \mathbf{B}^{\mathrm{bkg}} = B_0 \hat{e}_z$$

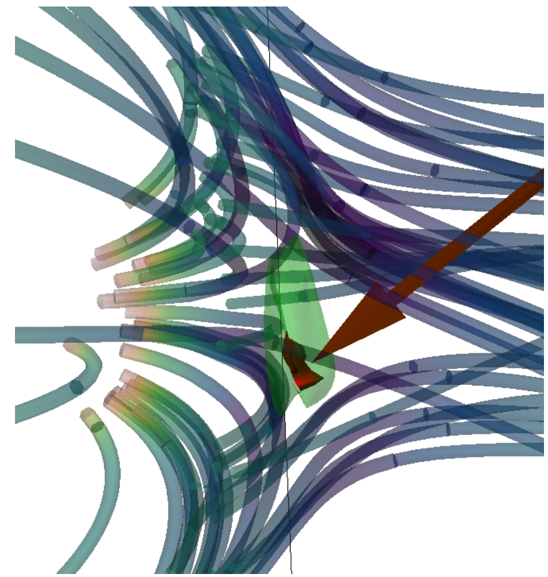
Magnetic Nulls

Singular current sheets observed at magnetic nulls (B = 0)

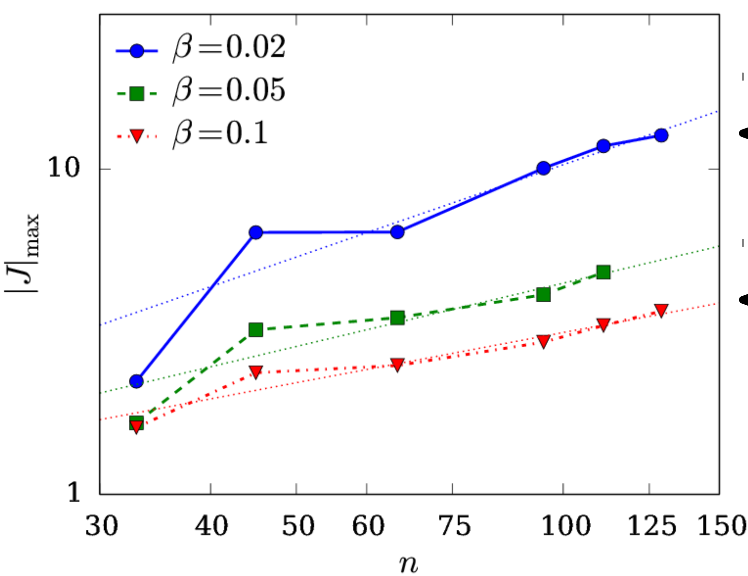
(Syrovatskiĭ 1971; Pontin & Craig 2005; Fuentes-Fernández & Parnell 2012, 2013; Craig & Pontin 2014)



$$\mathbf{u} = \mathbf{J} \times \mathbf{B}$$



$$\mathbf{u} = \mathbf{J} \times \mathbf{B} - \beta \nabla \rho$$



- singular current sheets at magnetic nulls
- Pressure cannot balance singularity.

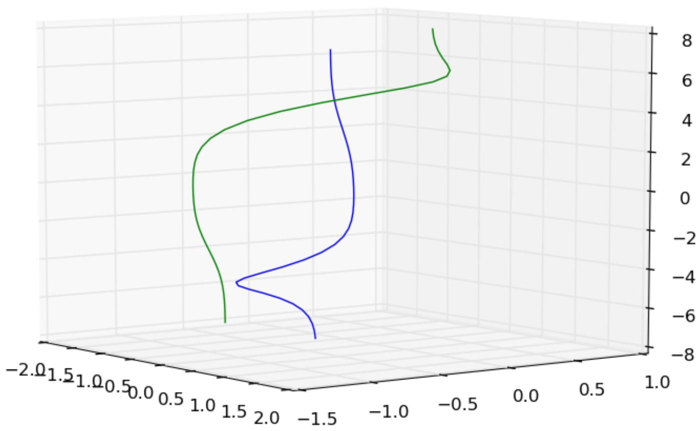
Code Details

- written in C++
- running in GPUs
- VTK data format
- 6^{th} order Runge-Kutta time stepping
- periodic and line-tied boundaries
- post processing routines in Python

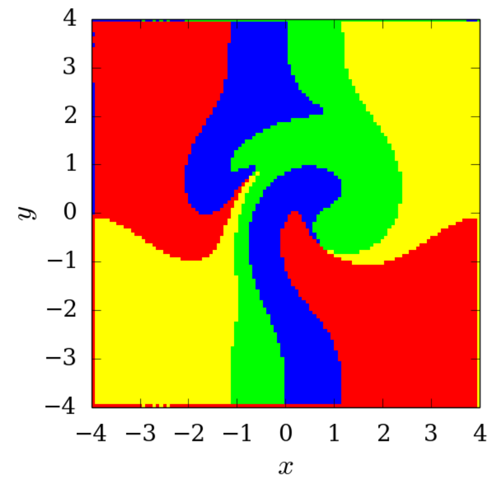
```
// compute the norm of JxB/B**2
__global__ void JxB_B2(REAL *B, REAL *J, REAL *JxB_B2, int dimX, int dimY, int dimZ) {
	int i = threadIdx.x + blockDim.x * blockIdx.x;
	int j = threadIdx.y + blockDim.y * blockIdx.y;
	int k = threadIdx.z + blockDim.z * blockIdx.z;
	int p = threadIdx.x;
	int q = threadIdx.y;
	int r = threadIdx.z;
	int l;
	REAL B2;

	// shared memory for faster communication, the size is assigned dynamically
	extern __shared__ REAL s[];
	REAL *Bs = s;                              // magnetic field
	REAL *Js = &s[3 * dimX * dimY * dimZ];     // electric current density
	REAL *JxBs = &Js[3 * dimX * dimY * dimZ];  // JxB

	// copy from global memory into shared memory
	if ((i < dev_p.nx) && (j < dev_p.ny) && (k < dev_p.nz)) {
		for (l = 0; l < 3; l++) {
			Bs[l + p*3 + q*dimX*3 + r*dimX*dimY*3] = B[l + (i+1)*3 + (j+1)*(dev_p.nx+2)*3 + (k+1)*(dev_p.nx+2)*(dev_p.ny+2)*3];
			Js[l + p*3 + q*dimX*3 + r*dimX*dimY*3] = J[l + i*3 + j*dev_p.nx*3 + k*dev_p.nx*dev_p.ny*3];
		}

		cross(&Js[0 + p*3 + q*dimX*3 + r*dimX*dimY*3],
			&Bs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3],
			&JxBs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3]);

		B2 = dot(&Bs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3], &Bs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3]);

		// return result into global memory
		JxB_B2[i + j*dev_p.nx + k*dev_p.nx*dev_p.ny] = norm(&JxBs[0 + p*3 + q*dimX*3 + r*dimX*dimY*3])/B2;
	}
}
```

Post-Processing

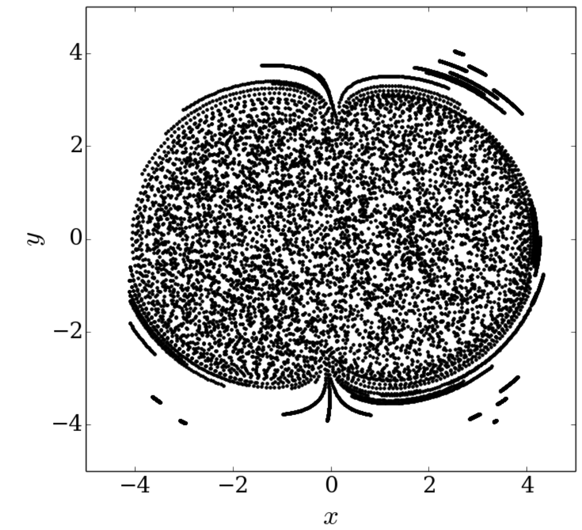
streamlines



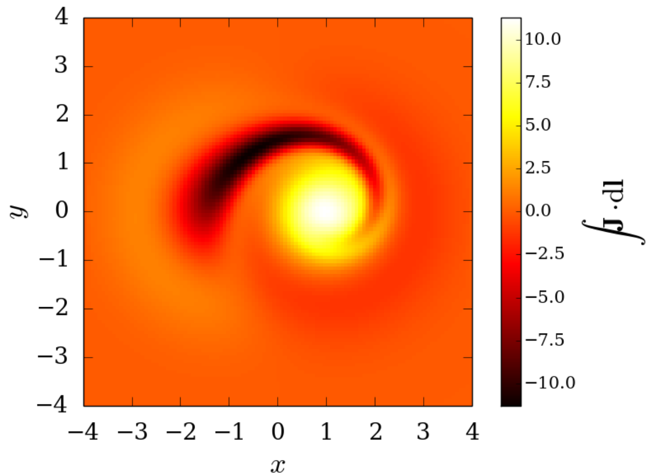
field line mapping



Poincaré maps



line integration



save and read as vtk file

```
s0 = gm.streamInit(tol = 0.01)
```

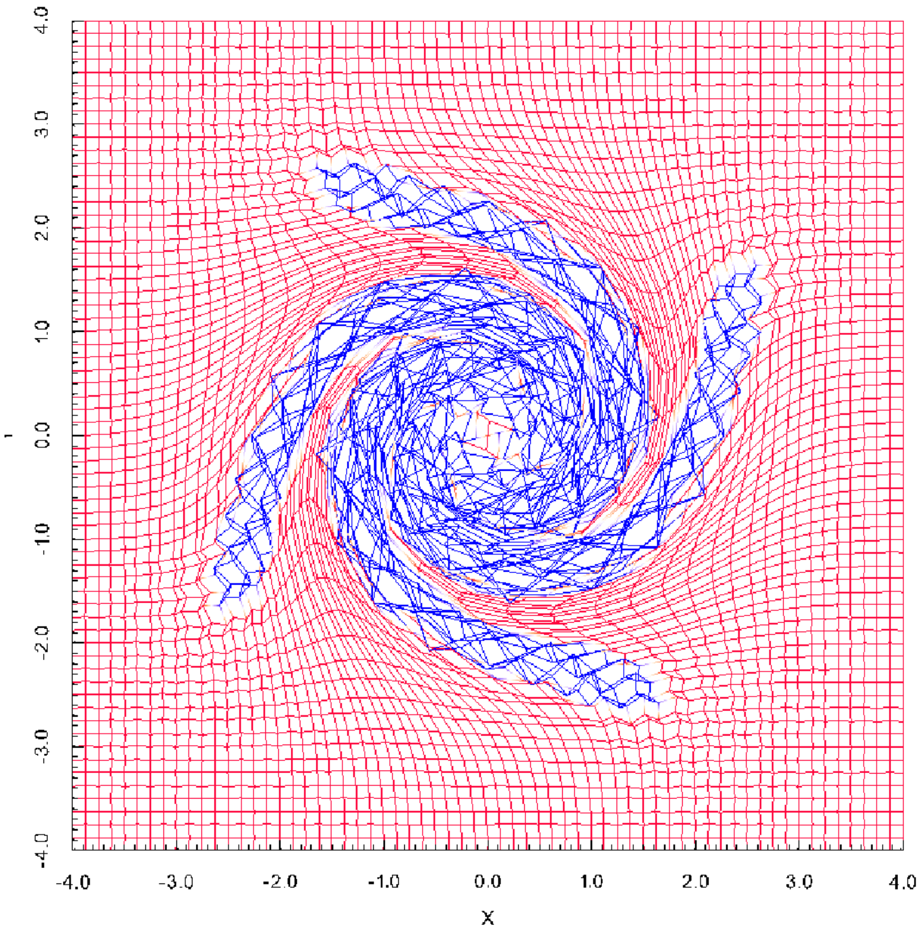
⇩

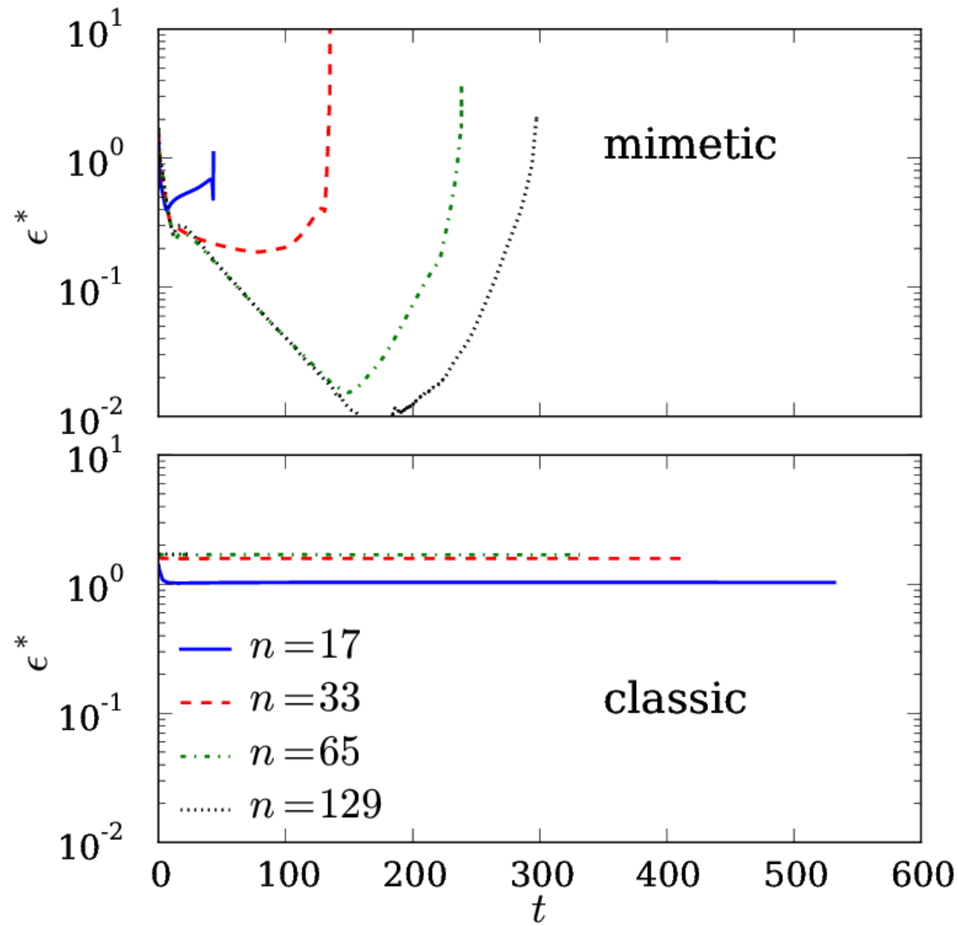
```
stream.vtk
```

⇩

```
sr = gm.readStream()
```

Limitations





red: convex
blue: concave

For concave cells the method becomes unstable.
But: results before crash better than classic method.