Quadratic helicity in MHD

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Quadratic magnetic helicities for thin magnetic tubes

The quadratic helicity $\chi^{(2)}$ is defined for a magnetic field **B** inside a bounded domain

$$\Omega \subset \mathbb{R}^3,$$

where **B** is tangent to the boundary of the domain. Assume, that magnetic field is represented by a (large) finite number of magnetic tubes Ω_i.

Quadratic magnetic helicities for thin magnetic tubes

The quadratic helicity $\chi^{(2)}$ is well-defined by the formula:

$$\chi^{(2)}(\mathbf{B}) = \sum_{i,j,k} \frac{\Phi_i^2 \Phi_j \Phi_k n(L_j, L_i) n(L_i, L_k)}{\mathrm{vol}(\Omega_i)}, \qquad (1)$$

where L_i, L_j, L_k is a non-ordered collection of central lines of the (thin) magnetic tubes Ω_i, Ω_j, Ω_k with magnetic fluxes Φ_i, Φ_j, Φ_k trough their cross-sections.

Quadratic magnetic helicities for thin magnetic tubes

In each collection of 3 tubes one magnetic tube Ω_i is singled out (marked). In equation (1) $n(L_j, L_i)$, $n(L_i, L_k)$ are pairwise linking coefficients of central lines of the corresponding magnetic tubes and $\mathrm{vol}(\Omega_i)$ are volumes filled by the magnetic tubes.
The value of formula (1) is not changed with respect to a subdivision of magnetic tubes into a collection of thinner tubes.

Quadratic magnetic helicities in MHD

Here we try to answer the following questions:

• Is it possible to use the quadratic helicity as non-linear restrictions in MHD problems?

• Is it possible to use the quadratic helicity as topological constraints in magnetic field relaxation?

The Arnold inequality

The following inequality:

$$U_{(2)}(\mathbf{B}) \geq C|\chi(\mathbf{B})|, \quad U_{(2)}(\mathbf{B}) = \int (\mathbf{B}, \mathbf{B}) \, d\Omega, \tag{2}$$

where $(.,.)$ denotes the scalar product, χ the magnetic helicity and C a positive constant, is called the Arnold inequality.
This inequality relates the magnetic energy (on the left hand side) to the magnetic helicity (on the right hand side). The constant $C > 0$ depends not on the magnetic field $\mathbf{B}$, but on the geometrical properties of the domain Ω, which is assumed to be a compact domain supporting $\mathbf{B}$.

A generalized Arnold inequality

The following inequality is satisfied:

$$\left(\frac{\pi}{16}\right)^{\frac{2}{3}} \mathrm{U}_{(6)}^{\frac{1}{3}}(\mathbf{B})(\mathrm{U}(\mathbf{B})_{(\frac{3}{2})})^{\frac{4}{3}} \geq \chi^{(2)}(\mathbf{B}) \geq \frac{\chi^2(\mathbf{B})}{2Vol(\Omega)}. \qquad (3)$$

The right hand side of the inequality (3) is an invariant under a smooth volume-preserved transformation of the domain Ω;

$$\mathrm{U}_{(k)} = \iiint |\mathbf{B}|^k d\Omega.$$

Quadratic magnetic helicity density

$$\chi^{(2)} = \lim_{a \to +\infty} \sum_{k=0}^{\infty} \frac{a^k}{k!} \iiint (\mathbf{A}, \mathbf{B}) \frac{d^{2k}(\mathbf{A}, \mathbf{B})}{d\tau^{2k}} d\Omega, \tag{4}$$

where τ is the magnetic parameter on magnetic lines, in particular,

$$\frac{d(\mathbf{A}, \mathbf{B})}{d\tau} = (\mathbf{grad}(\mathbf{A}, \mathbf{B}), \mathbf{B}). \tag{5}$$

Example

Assume that the magnetic field contains the closed magnetic line L. The cyclic covering over this magnetic line is the real line $\tilde{L} = (-\infty, +\infty)$ with the period 2π. Assume that $(\mathbf{A}, \mathbf{B}) = \bar{h} + \sin(\tau)$, $\tau \in \tilde{L}$. Then the first term of the integral (4) is $\bar{h}^2 + \frac{1}{2}$, the second term is $\frac{1}{2}$ and the quadratic helicity over L equals to $\bar{h}^2$.

Example

For last terms in the formula (5) we get:

$$\int_0^{2\pi} (\mathbf{A},\mathbf{B}) \frac{d^{2k}(\mathbf{A},\mathbf{B})}{d\tau^{2k}}\, d\tau = \frac{(-1)^k}{2\pi} \int_0^{2\pi} \sin^2(\tau_2)\, d\tau_2,$$

$$\frac{1}{2\pi} \int_0^{2\pi} \sin^2(\tau_2)\, d\tau_2 = \frac{1}{2},$$

$$\chi^{(2)} = \bar{h}^2 + \frac{1}{2} + \lim_{a \to +\infty} \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k a^k}{k!} = \bar{h}^2.$$

Compressibility effects on the quadratic helicity $\chi^{(2)}$

Use the square of the average magnetic helicity along a magnetic line $\Lambda^{(2)}(T;\mathbf{x})$. It is defined as

$$\Lambda^{(2)}(T;\mathbf{x}) = \frac{1}{T^2}\left(\int_0^T (\dot{\mathbf{x}}(\tau), \mathbf{A}(\mathbf{x}(\tau)))\ d\tau\right)^2, \quad (6)$$

with the magnetic vector potential $\mathbf{A}$ of the field and the velocity vector $\dot{\mathbf{x}}(\tau) = \mathbf{B}(\mathbf{x}(\tau))$. The integral is taken along a magnetic field line starting at position $\mathbf{x} = \mathbf{x}(\tau = 0)$.

Compressibility effects on the quadratic helicity $\chi^{(2)}$

The quadratic helicity $\chi^{(2)}$ is

$$\chi^{(2)} = \limsup_{T \to \infty} \int \Lambda^{(2)}(T; \mathbf{x})\ dD, \qquad (7)$$

where $D \in \mathbb{R}^3$ is a ball of radius r. By Birkhoff Theorem the limit in (6) exists for almost arbitrary $\mathbf{x}$. The velocity $\dot{\mathbf{x}}(\tau)$ along the magnetic field line is equivalent to $\mathbf{B}$ at position $\mathbf{x}(\tau)$.

Compressibility effects on the quadratic helicity $\chi^{(2)}$

We now investigate the cases of diffeomorphisms stretching the coordinate system along and across the magnetic field lines assuming a fluid density of $\rho_0 = 1$ before applying the mapping. For a stretching along the magnetic field lines by a factor of λ, the density changes to $\rho = \lambda^{-1}$. **B** is invariant under such a transformation because the integral magnetic flow is invariant and the cross-section is fixed.

Compressibility effects on the quadratic helicity $\chi^{(2)}$

The total length of the field line at parameter $\tau = T$ changes, but the mean of the magnetic helicity density does not, hence $\overline{(\mathbf{A},\mathbf{B})} \mapsto \overline{(\mathbf{A},\mathbf{B})\rho^{-1}}$. So, the function $\overline{(\mathbf{A},\mathbf{B})\rho^{-1}}$ is frozen in. We now substitute $(\mathbf{A},\mathbf{B})$ by $(\mathbf{A},\mathbf{B})\rho^{-1}$ in equation (6) which adds a factor of ρ^{-1} in equation (7):

$$\Lambda_{\rho}^{(2)}(T;\mathbf{x}) = \frac{1}{T^2}\left(\int_0^T \frac{(\dot{\mathbf{x}}(\tau),\mathbf{A}(\mathbf{x}(\tau)))}{\rho(\mathbf{x}(\tau)))}\, d\tau\right)^2. \tag{8}$$

However, with ρ also the measure dD changes to ρdD. We get the following formula for $\chi^{(2)}$ in the compressed fluid:

$$\chi_{\rho}^{(2)} = \iint \Lambda_{\rho}^{2}\rho \; dD. \tag{9}$$

Compressibility effects on the quadratic helicity $\chi^{(2)}$

For a stretching across the magnetic field lines by a factor of λ along both directions the density changes as $\rho = \lambda^{-2}$. In order to conserve magnetic flux across comoving surfaces the magnetic field changes according to $\lambda^{-2}\mathbf{B} = \rho\mathbf{B}$. Again, $\overline{(\mathbf{A}, \mathbf{B})\rho^{-1}}$ is invariant. The integral measure changes according to λ^{-2}.

In both cases the additional factor of ρ in the two integrands cancel if ρ does not depend on space. In that case we obtain a quadratic magnetic helicity that is invariant under (homogeneously) density changing diffeomorphisms.

Compressibility effects on the quadratic helicity $\chi^{(2)}$

From the changes of $\chi^{(2)}$ for general transformations, we can conclude that $\chi^{(2)}$ is not a function of the magnetic helicity χ. If it was true, then for two fields with the same helicity, the quadratic helicity would be the same. Here we construct a simple counter example. Take a field $\mathbf{B}_1$ with helicity χ_1. Construct a second field $\mathbf{B}_2$ via a topology conserving transformation of $\mathbf{B}_1$, then $\chi_1 = \chi_2$. If this transformation is not volume conserving, then, in general, the quadratic helicities are different, i.e. $\chi_1 \neq \chi_2$.

Magnetic force-free configuration

Let $P \subset \Lambda^2$ be the right triangle (all 3 vertexes on the absolute) on the Lobachevskii plane Λ^2, $f : P \to S^2_+$ be the conformal transformation (the Picard analytic function) of the square onto the upper semi-sphere of the Rimannian sphere S^2. The vertexes a, b, c of P are mapped into points $f(a), f(b), f(c)$ at the equator $S^1 \subset S^2$ and we assume that $dist(f(a), f(b)) = dist(f(b), f(c)) = dist(f(c), f(d)) = \frac{2\pi}{3}$.

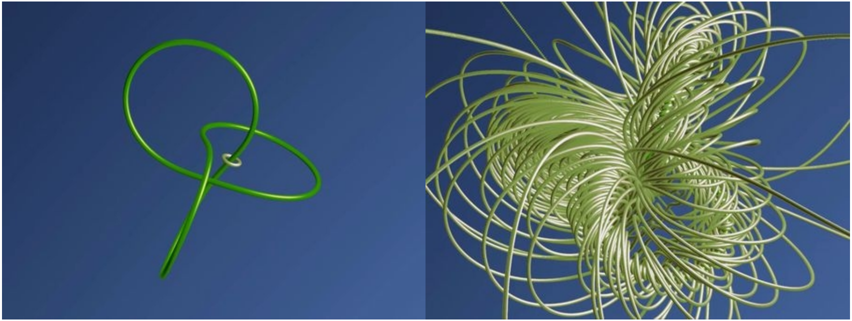
Magnetic force-free configuration

Denote by $g : \Lambda^2 \to S^2$ the branched cover with ramifications at $f(a), f(b), f(c)$, which is defined as the conformal periodic extension of f on the Lobachevskii plane. The unite tangent bundle $T(\Lambda^2)$ is equipped with the geodesic flows $\mathbf{B}_{left}$ on Λ is factorized by the monodromy group of the covering g. The mapping $Tg : T\Lambda^2 \to TS^2$ induces the magnetic flow on the Lee group $SO(3)$ with a non-homogeneous Riemannian metric, which is conformal equivalent to the standard metric by the scalar factor $\hat{\rho}$. This factor is the density ρ on the standard sphere S^3, which is the double covering over $SO(3)$.

Magnetic force-free configuration

• A magnetic force-free field $\mathbf{B}_{left}$ (in the standard non-homogeneous sphere (S^3, rho) with the total finite volume $Vol = \int \rho dS^3$) with the finite magnetic energy is well-defined as the pull-back by a 12-sheeted branching covering over the Lorenz attractor by *Etienne Ghys and Jos Leys*.

Magnetic force-free configuration



This covering transforms the standard 3-component Hopf link $L \subset S^3$ into the exceptional trefoil of the Lorenz attractor.

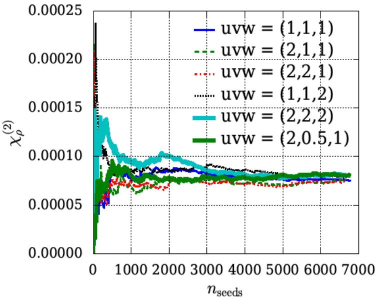
Magnetic force-free configuration

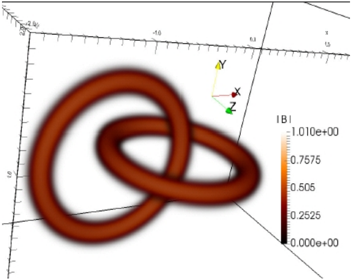
• The density function ρ in S^3 has an asymptotic $\exp(z^{-1})$ near L, where z is the distance to L (a huge extremal over the magnetic pinch).
• Quadratic helicity $\chi^{(2)}$ of $\mathbf{B}_{left}$ takes the minimal possible value

$$\chi^{(2)} = \frac{\chi^2}{2Vol}.$$

• The stereographic projection $S^3 \setminus pt \to \mathbb{R}^3$ transforms $\mathbf{B}_{left}$ into a force-free magnetic field with a finite magnetic energy in non-homogeneous space $\mathbb{R}^3$.

Numerical calculations of $\chi^{(2)}$





Quadratic Magnetic Helicity Flow

The quadratic magnetic helicity $\chi^{(2)}$ admits a continuous variation in the case of C^2–small flows (vector of flows $\frac{\partial \mathbf{B}}{\partial t}$ in the domain Ω with its first and second partial derivatives small).
In particular, for α^2-dynamos we get: $\frac{\partial \mathbf{A}}{\partial t} = \alpha \mathbf{B}$; and, assuming $\mathrm{rot}\mathbf{B} = k\mathbf{B}$, we have: $\frac{d\chi^{(2)}}{dt} = 4\alpha k \chi^{(2)}$. This means that the flow of quadratic magnetic helicity $\chi^{(2)}$ for the magnetic field with the **k**-vector coincides with the flow of the square of the helicity χ^2.

Conclusions

• We argue that higher helicity invariants have several properties analogous to Gauss invariant, which allows to constrain dynamo action. Quadratic helicity $\chi^{(2)}$ is an invariant of the ideal MHD.

• The quadratic helicity of a force-free field with a constant magnetic helicity density satisfies the lower bound in the generalized Arnold inequality.

Reference

Calculations for the practical applications of quadratic helicity in MHD
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