Mechanism of mean flow generation in rotating turbulence and its relation to inertial wave

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Helicity distribution in rotating sphere

◦ Duarte *et al.*, MNRAS (2016)
helicity distribution in compressible MHD turbulence

- anti-symmetric sign in north and south
  $\longrightarrow \alpha$ effect generates the polar field
- radially inhomogeneous

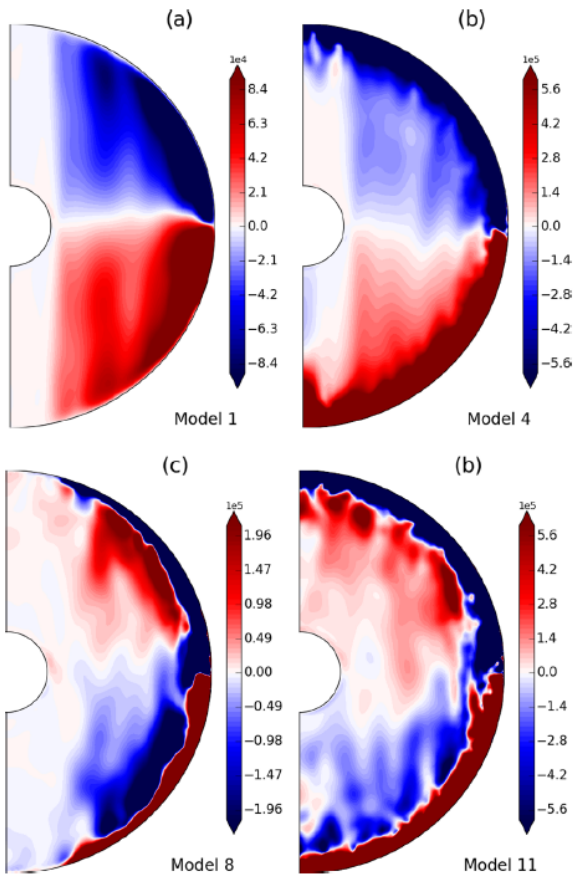


Figure 5. Azimuthally averaged contour plots of kinetic helicity. The top row displays the helicity pattern described in Section 2.1 for models 1 (a) and 4 (b) of Table 1. The bottom row shows two examples of the regime described in Section 2.2 for cases 8 (c) and 11 (d) of the same table.

How about the effects of helicity on velocity field?

Effect of helicity on the energy cascade

- André and Lesieur, JFM (1977)
  $\longrightarrow$ helicity does not affect the energy cascade in fully developed turbulence
- Rogers and Moin, PoF (1987), Wallace *et al.*, PoF (1992)
  $\longrightarrow$ helicity does not correlate with small-dissipation events
- Stepanov *et al.*, PRL (2015)
  inertial range spectrum

$$E(k) \sim \varepsilon^{2/3} k^{-5/3} (1 - |H^r(k)|)^{-1/3}, \quad H^r(k) = \frac{H(k)}{2kE(k)},$$

$$\frac{1}{2} \left\langle u_i' u_i' \right\rangle = \int dk \; E(k), \quad \left\langle u_i' \omega_i' \right\rangle = \int dk \; H(k)$$

hindering effect requires high relative helicity

Effect of helicity on momentum transport

- Yokoi and Yoshizawa, PoF (1993); RANS model

$$\frac{\partial U_i}{\partial t} = -\frac{\partial}{\partial x_j} U_i U_j - \frac{\partial R_{ij}}{\partial x_j} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} + 2\epsilon_{ij\ell} U_j \Omega_\ell^F$$

$$R_{ij} = \frac{2}{3} K \delta_{ij} - 2\nu_T S_{ij} + \eta \left(\frac{\partial H}{\partial x_j} \Omega_i^A + \frac{\partial H}{\partial x_i} \Omega_j^A - \frac{2}{3} \delta_{ij} \frac{\partial H}{\partial x_\ell} \Omega_\ell^A \right)$$

$q = (\boldsymbol{u}, p, \boldsymbol{\omega}), \quad q = Q + q', \quad Q = \langle q \rangle$
$R_{ij} = \langle u_i' u_j' \rangle$: Reynolds stress
$H = \langle u_i' \omega_i' \rangle$: turbulent helicity, $\boldsymbol{\Omega}^A = \boldsymbol{\nabla} \times \boldsymbol{U} + 2\boldsymbol{\Omega}^F$

zero-mean velocity and non-zero helicity and rotation;
$\boldsymbol{U} = 0, \quad \boldsymbol{\nabla} H, \boldsymbol{\Omega}^F \neq 0$

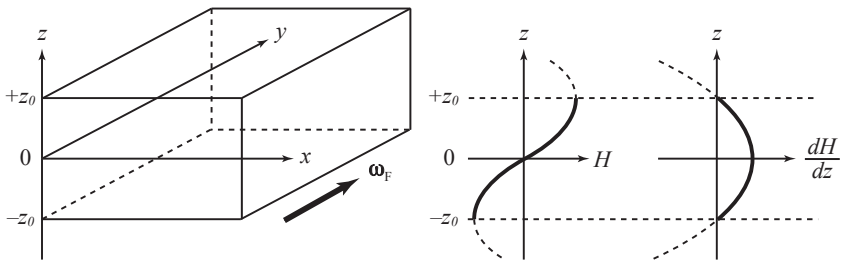
$$R_{ij} - \frac{2}{3} K \delta_{ij} = 2\eta \left(\frac{\partial H}{\partial x_j} \Omega_i^F + \frac{\partial H}{\partial x_i} \Omega_j^F - \frac{2}{3} \delta_{ij} \frac{\partial H}{\partial x_\ell} \Omega_\ell^F \right) \neq 0,$$

<u>mean-flow generation from zero-mean velocity</u>

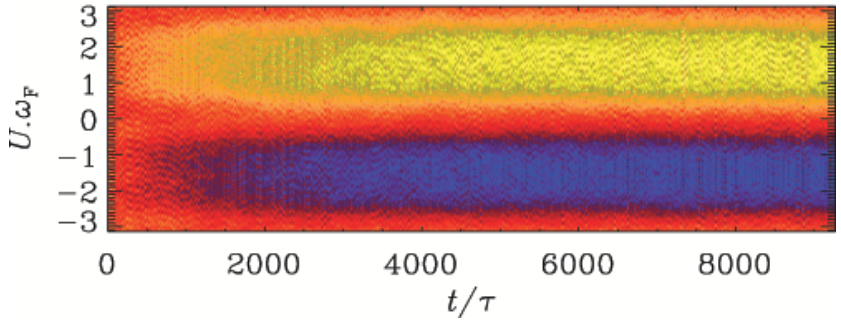
(Yokoi and Brandenburg, PRE (2016))

Mean-flow generation

- Yokoi and Brandenburg, PRE (2016)
starting with zero-mean velocity



Flow configuration



Evolution of axial mean velocity

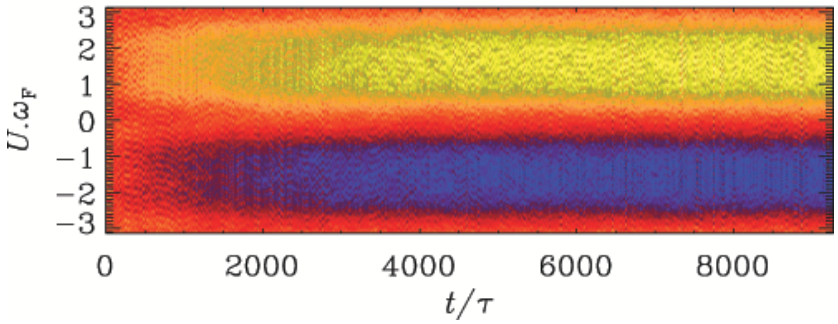
$$\boldsymbol{\Omega}^F = (0, \Omega^F, 0),\ H(z) \propto \sin(\pi z/z_0)$$

$$R_{yz} = \eta \frac{\partial H}{\partial z} 2\Omega^F \neq 0 \quad \Rightarrow \quad \frac{\partial U_y}{\partial t} = -\frac{\partial R_{yz}}{\partial z} \neq 0$$

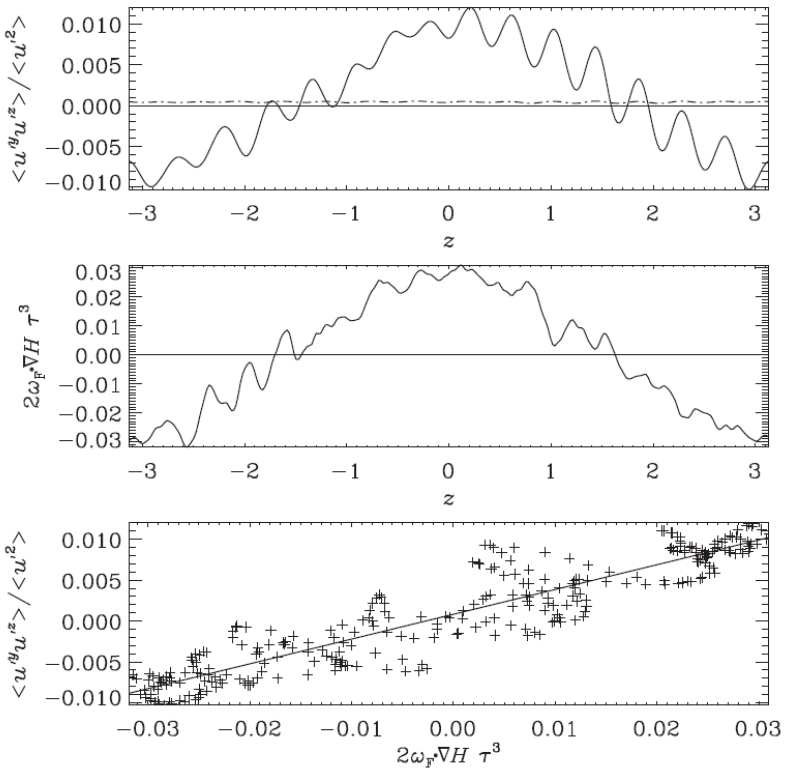
- Positive (negative) axial mean velocity is generated around positively (negatively) helical region

Distribution of the Reynolds stress

- Yokoi and Brandenburg, PRE (2016)



Evolution of axial mean velocity



top right: Reynolds stress
middle: model expression

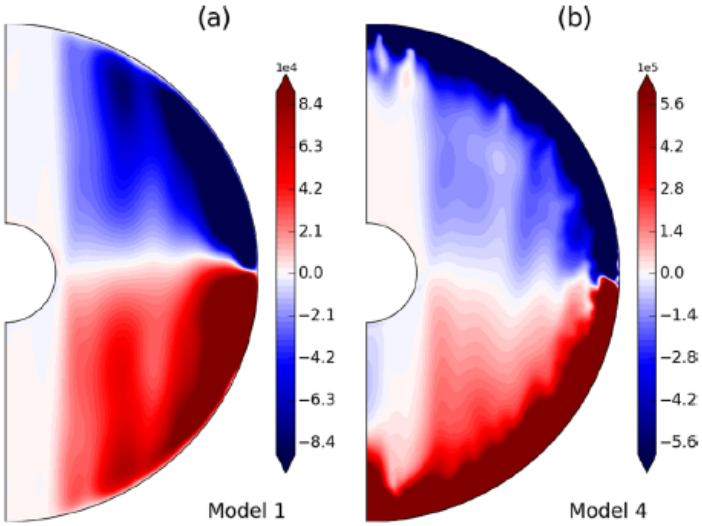
$$R_{yz} = \eta \frac{\partial H}{\partial z} 2\Omega^F$$

bottom: correlation

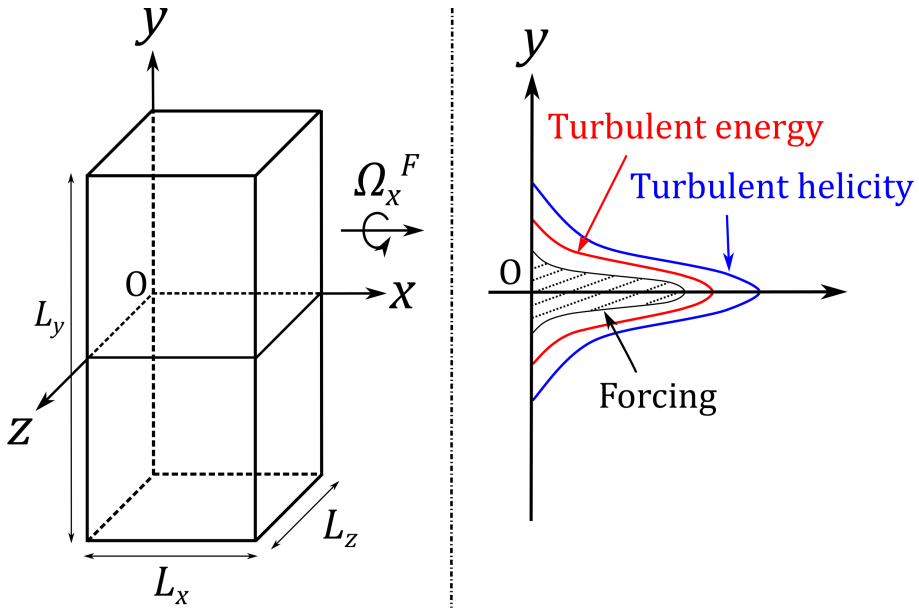
Correlation between the Reynolds stress and the model in early stage

Q. How helicity affects the Reynolds stress?

Inhomogeneity of helicity in rotating sphere



Duarte *et al*. (2016)



inhomogeneous direction of helicity is perpendicular to the rotation axis in the low-latitude region

Numerical simulations

Large-eddy simulation (LES)

$$\frac{\partial \overline{u}_i}{\partial t} = -\frac{\partial}{\partial x_j}\overline{u}_i\overline{u}_j - \frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\nu_{sgs}\overline{s}_{ij}\right) + 2\epsilon_{ij\ell}\overline{u}_j\Omega^F_\ell + \overline{f}_i$$

Smagorinsky model:

$\nu_{sgs} = (C_S\Delta)^2\sqrt{2\overline{s}_{ij}\overline{s}_{ij}}$

$\Delta = (\Delta x\Delta y\Delta z)^{1/3},\ C_S = 0.19$

$\overline{q}$: grid-scale variables

periodic boundary condition

grid size: $128 \times 256 \times 128$

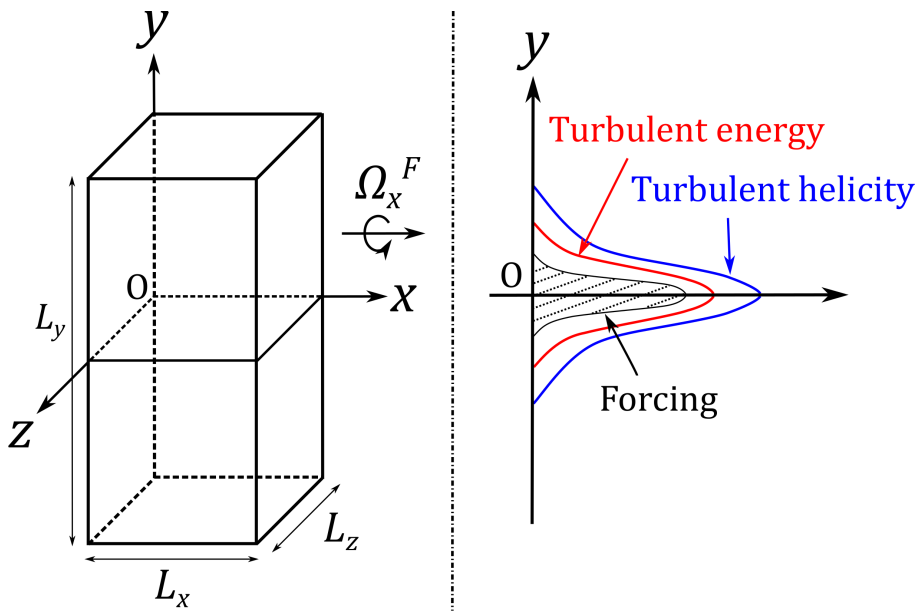
box size: $2\pi \times 4\pi \times 2\pi$

2nd order finite difference

Adams–Bashforth method

initial mean velocity: $\langle \overline{u}_i \rangle_S = 0$

Inagaki *et al.*, PRFluids (2017)



$\mathbf{\Omega}^F = (\Omega^F, 0, 0),\ H^{GS}(y)$

Run 1: non-helical, non-rotating

Run 2: helical, non-rotating

Run 3: non-helical, rotating

Run 4: helical, rotating

Numerical simulations

Forcing spectrum

$E^{ex}(k) = k^{-5/3}$ for $10 \leq k \leq 14$,

$E_H^{ex}(k) = 2\alpha k E^{ex}(k),\ |\alpha| \leq 1$

$\langle \overline{u}_i' \overline{u}_i' \rangle_S (y=0)/2 = 1$

$\langle \overline{f}_i \rangle_S = 0$

Rossby number

$$\mathrm{Ro} = \frac{\varepsilon^{SGS}/K^{GS}}{2\Omega^F}(y=0)$$

Run 1: $\alpha = 0, \quad \mathrm{Ro} = \infty$

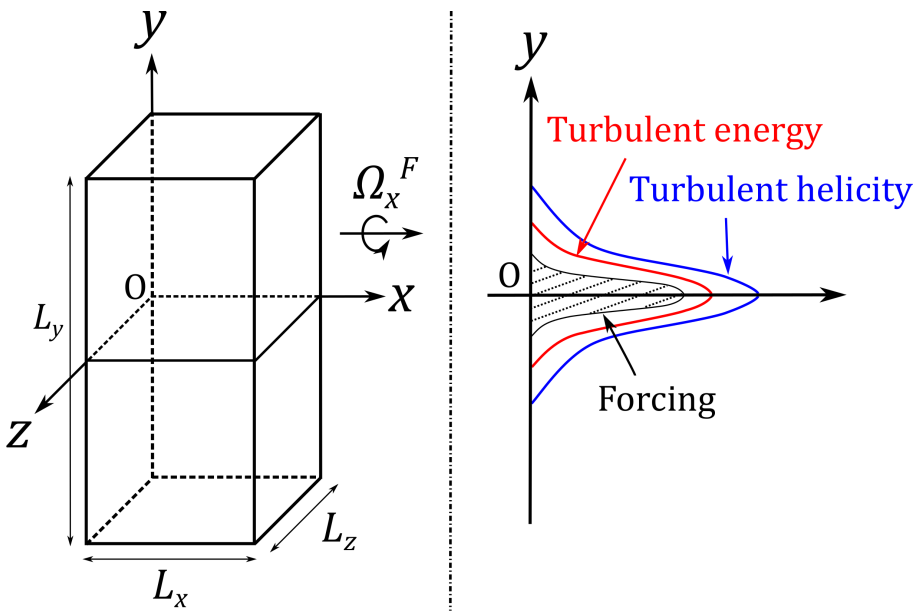
Run 2: $\alpha = 0.5, \mathrm{Ro} = \infty$

Run 3: $\alpha = 0, \quad \mathrm{Ro} = 0.185$

Run 4: $\alpha = 0.5, \mathrm{Ro} = 0.166$

initial mean velocity: $\langle \overline{u}_i \rangle_S = 0$

Inagaki *et al.*, PRFluids (2017)



$\mathbf{\Omega}^F = (\Omega^F, 0, 0),\ H^{GS}(y)$

Run 1: non-helical, non-rotating

Run 2: helical, non-rotating

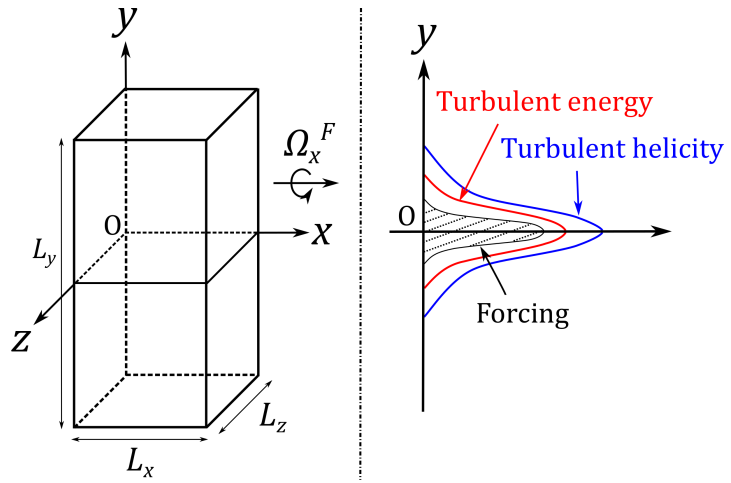
Run 3: non-helical, rotating

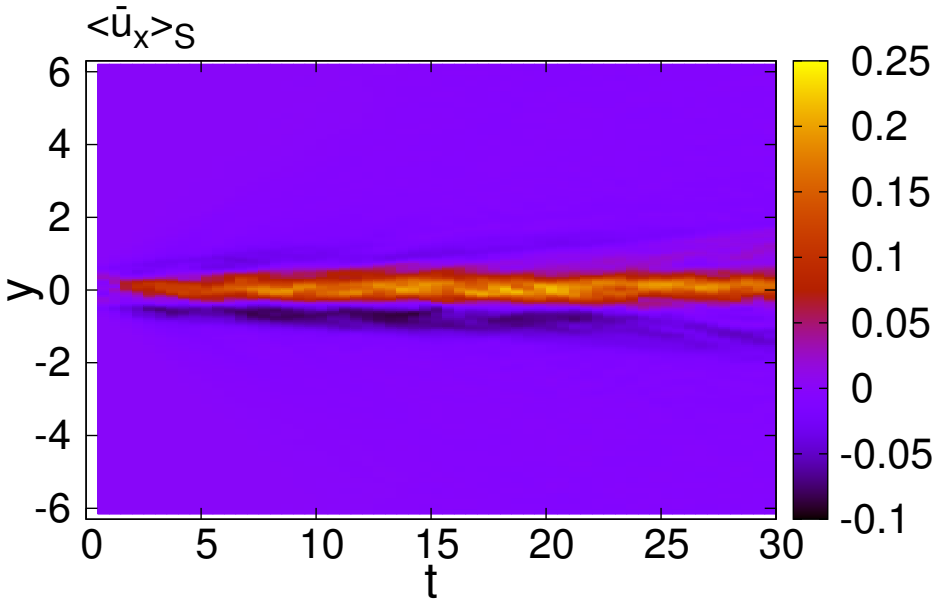
Run 4: helical, rotating

Generation of axial mean velocity

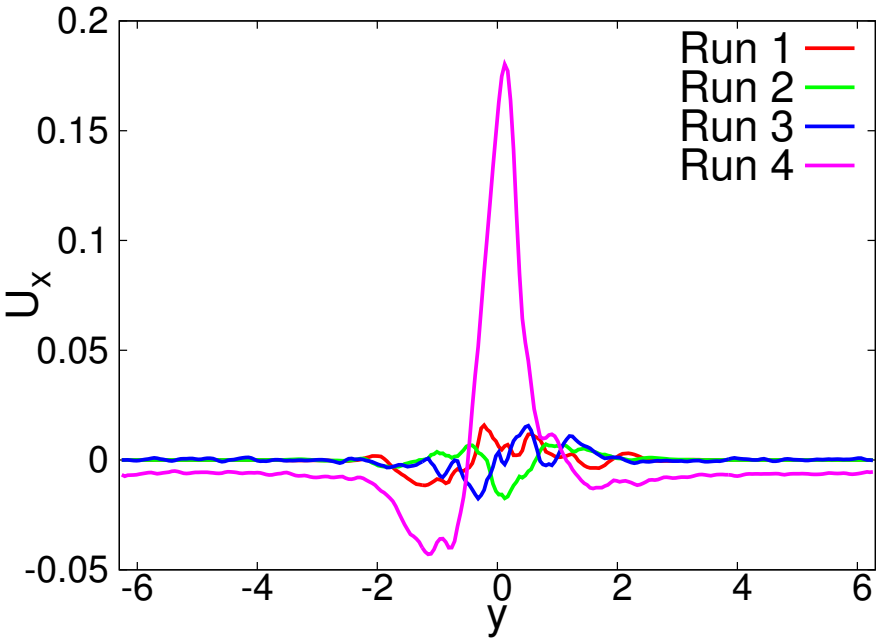
Run 1: non-helical, non-rotating
Run 2: helical, non-rotating
Run 3: non-helical, rotating
Run 4: helical, rotating

Time average over $20 \leq t \leq 30$





Time evolution of the axial mean velocity for Run 4



Time averaged axial mean velocity for each run

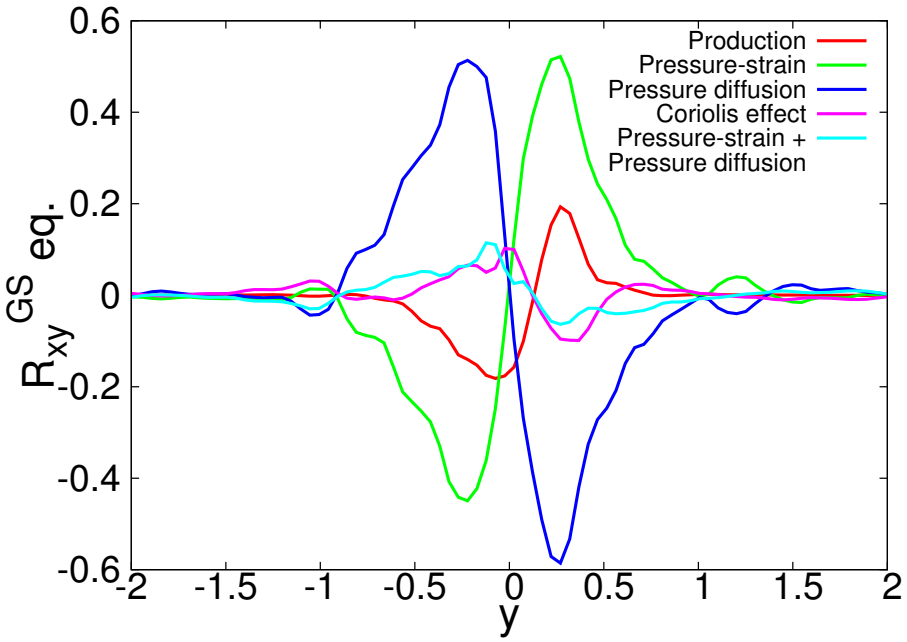
Reynolds stress transport

$$\frac{\partial U_x}{\partial t} = \underline{-\frac{\partial R_{xy}^{GS}}{\partial y}} + \frac{\partial}{\partial y}\left\langle \nu_{sgs}\overline{s}_{xy}\right\rangle$$

$$\frac{\partial R_{xy}^{GS}}{\partial t} = \underline{-R_{yy}^{GS}\frac{\partial U_x}{\partial y}} \underline{+2\left\langle \overline{p}'\overline{s}'_{xy}\right\rangle} \underline{-\frac{\partial}{\partial y}\left\langle \overline{p}'\overline{u}'_x\right\rangle} \underline{+2R_{xz}^{GS}\Omega^F}$$

$R_{ij}^{GS} = \langle \overline{u}'_i\overline{u}'_j\rangle$

- positive gradient around $y = 0$
  $\Rightarrow$ destruct the mean flow (eddy viscosity)
- negative gradient around $y = 0$
  $\Rightarrow$ enhance the mean flow



Budget of R_{xy}^{GS} eq. for Run 4

focus on the pressure diffusion

Relationship between the pressure diffusion and helicity

Poisson equation for the pressure

$$\nabla^2 \overline{p}' = -2\overline{s}'_{ab} S_{ab} + \overline{\omega}'_a \Omega^A_a - \overline{s}'_{ab}\overline{s}'_{ab} + \frac{1}{2}\overline{\omega}'_a \overline{\omega}'_a + \frac{\partial^2}{\partial x_a \partial x_b}\left(2\nu_{sgs}\overline{s}_{ab}\right)'$$

$$\nabla^2 \overline{p}' \simeq -\ell_p^{-2}\,\overline{p}'$$

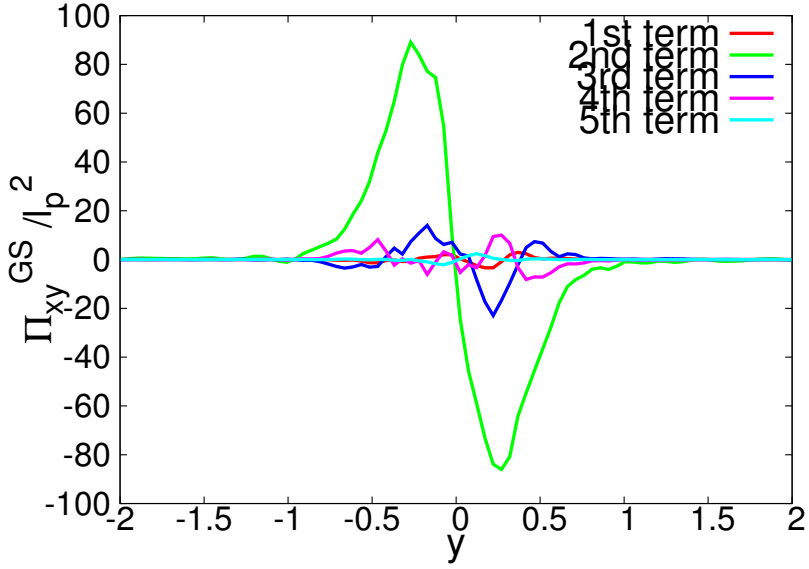
$$-\frac{\partial}{\partial y}\left\langle \overline{p}'\overline{u}'_x \right\rangle = \ell_p^2 \frac{\partial}{\partial y}\Big[-2\left\langle \overline{u}'_x \overline{s}'_{ab}\right\rangle S_{ab} \underline{+ \left\langle \overline{u}'_x \overline{\omega}'_a \right\rangle \Omega^A_a} - \left\langle \overline{u}'_x \overline{s}'_{ab}\overline{s}'_{ab}\right\rangle$$

$$+\frac{1}{2}\left\langle \overline{u}'_x \overline{\omega}'_a \overline{\omega}'_a \right\rangle + \left\langle \overline{u}'_x \frac{\partial^2}{\partial x_a \partial x_b}\left(2\nu_{sgs}\overline{s}_{ab}\right)\right\rangle\Big]$$

$$-\frac{\partial}{\partial y}\left\langle \overline{p}'\overline{u}'_x \right\rangle \simeq \frac{\partial}{\partial y}\left(2\ell_p^2 \underline{\left\langle \overline{u}'_x \overline{\omega}'_x \right\rangle} \Omega^F_x\right)$$

$$\simeq \frac{\partial}{\partial y}\left(\frac{2}{3}\ell_p^2 \underline{H^{GS}} \Omega^F_x\right)$$

$$H^{GS} = \left\langle \overline{u}'_i \overline{\omega}'_i \right\rangle, \left\langle \overline{u}'_i \overline{\omega}'_j \right\rangle = H^{GS}\delta_{ij}/3$$



Approximate estimation of the pressure diffusion for Run 4

Model expression for the Reynolds stress

Pressure diffusion

$$\Pi_{ij} = -\frac{\partial}{\partial x_j}\left\langle p' u_i' \right\rangle + (i \leftrightarrow j) = \frac{\partial}{\partial x_j}\left(\ell_p^2 H \Omega_i^A\right) + (i \leftrightarrow j)$$

Reynolds stress transport equation* $\quad B_{ij} = R_{ij} - (2/3)K\delta_{ij}$

$$\frac{\partial B_{ij}}{\partial t} = -\frac{B_{ij}}{\tau} + \frac{4}{3}KS_{ij} + \underline{\Pi_{ij}} + \ldots$$

$$\Rightarrow B_{ij} = -2\nu_T S_{ij} + \tau\left[\frac{\partial}{\partial x_j}\left(\ell_p^2 H \Omega_i^A\right) + \frac{\partial}{\partial x_i}\left(\ell_p^2 H \Omega_j^A\right)\right]_D$$

similar to the model of Yokoi and Yoshizawa (1993)

$$B_{ij} = -2\nu_T S_{ij} + \eta\left[\frac{\partial H}{\partial x_j}\Omega_i^A + \frac{\partial H}{\partial x_i}\Omega_j^A\right]_D$$

$[A_{ij}]_D = A_{ij} - A_{\ell\ell}\delta_{ij}/3$

* Rotta (1951), Launder *et al.* (1975)

Physical meaning of the pressure flux through helicity

Turbulent kinetic energy equation $K = \langle u_i' u_i' \rangle / 2$

$$\frac{DK}{Dt} = -R_{ij} S_{ij} - \varepsilon - \frac{\partial}{\partial x_i} \left[\left\langle u_i' \frac{1}{2} u_j' u_j' \right\rangle + \langle u_i' p' \rangle - \nu \frac{\partial K}{\partial x_i} \right]$$

$\langle u_i' p' \rangle$: energy flux of x_i direction due to pressure (pressure flux)

model expression using helicity

$$\left\langle u_i' p' \right\rangle = -2 \ell_p^2 H \Omega_i^F$$

- negative (positive) helicity corresponds to positive (negative) axial energy flux

inertial wave; group velocity $\boldsymbol{C}^g$ (cf. Davidson (2004))

$$C_i^g = \pm \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \frac{2\Omega_j^F}{k} \sim \pm \frac{2\Omega_i^F}{k}, \quad \tilde{u}_i \tilde{\omega}_i^* = \mp k |\tilde{u}_i|^2$$

- negative (positive) helicity corresponds to positive (negative) axial wave packet propagation

Relationship between the pressure flux and inertial wave

Homogeneity is assumed

$$\left\langle u_i' p' \right\rangle |_{\Omega^F} = -\int d^3k \; D_{ij}(\mathbf{k}) \frac{h(\mathbf{k})}{2k} \frac{2\Omega_j^F}{k}, \;\; \left\langle u_i' \omega_i' \right\rangle = \int d^3k \; h(\mathbf{k})$$

$D_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$

statistical property of inertial wave

$$\tilde{u}_i \tilde{\omega}_i^* = \mp k |\tilde{u}_i|^2 \;\Rightarrow\; e(\mathbf{k}) = \mp \frac{h(\mathbf{k})}{2k}, \;\; \frac{1}{2} \left\langle u_i' u_i' \right\rangle = \int d^3k \; e(\mathbf{k}),$$

$$C_i^g = \pm D_{ij}(\mathbf{k}) \frac{2\Omega_j^F}{k}$$

$$\Rightarrow \; \left\langle u_i' p' \right\rangle |_{\Omega^F} = \int d^3k \; e(\mathbf{k}) C_i^g$$

energy flux due to inertial wave

Conclusions

- Axial mean velocity was generated and sustained stating from the zero-mean-velocity configuration
- Mean-flow generation was only observed when both the inhomogeneous helicity and the system rotation exist
- The pressure diffusion significantly contributes to the sustainment of the emerged mean flow.
- The pressure diffusion was expressed as the gradient of turbulent helicity coupled with the angular velocity of the system rotation
- It was shown that energy flux due to the pressure is closely related to the group velocity of inertial wave

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