Seismic diagnosis from gravity modes strongly affected by rotation

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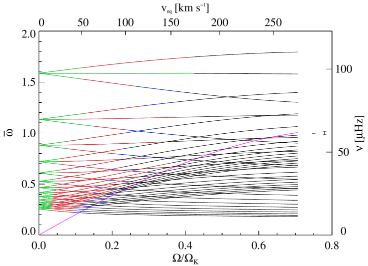
Effect of rotation on stellar oscillation modes

Slow rotators

- perturbative approach → rotational splittings
- core/envelope rotation contrast (Beck et al., 2012; Mosser et al., 2012; Deheuvels et al., 2012, 2014, 2015)

Fast rotators: poor understanding

- no longer spherical
- perturbative methods no longer valid
- typical intermediate-mass and massive stars are concerned (e.g. δ-Scuti, γ-Doradus)
- validity of the traditional approximation?



Ballot et al. (2010)

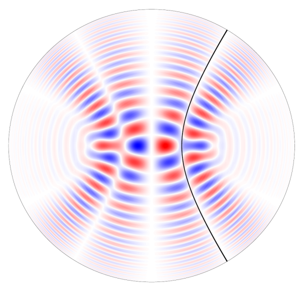
Two solutions

- full-2D numerical computations (expensive, limited physical insight)
- asymptotic theories (approximate, but more insightful)

Asymptotic theories?

Acoustic modes (Lignières & Georgeot, 2009; Pasek et al., 2012)

- short-wavelength approximation of the wave equation
- ray trajectories studied as Hamiltonian systems
- modes from positive interference of rays
- regular patterns predicted in oscillation spectra
- successfully confronted to numerically computed modes



Pasek et al. (2012)

Gravito-inertial modes (Prat et al., 2016)

- general eikonal equation including Coriolis and centrifugal effects
- ray dynamics $\rightarrow$ exploration of the mode properties
- next step: provide diagnosis tools

Outline

1. Summary of the results of Prat et al. (2016)
2. Seismic diagnoses

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   - General eikonal equation
   - Exploration of the ray dynamics
2. Seismic diagnoses

Wave equation for axisymmetric modes

Assumptions

- polytropic model of star
- uniform rotation
- adiabatic and inviscid oscillations
- Cowling approximation: perturbations of the gravitational potential neglected

A deceptively simple equation

$$\Delta\hat{\Psi} = \underbrace{-\frac{\omega^2}{{c_s}^2}\hat{\Psi}}_{\text{acoustic waves}} + \underbrace{\frac{f^2}{\omega^2}\Delta_z\hat{\Psi}}_{\text{inertial waves}} + \underbrace{\frac{{N_0}^2}{\omega^2}\Delta_\perp\hat{\Psi}}_{\text{gravity waves}} + C\hat{\Psi}$$

- C is very (very) complicated
- complex geometry because of the centrifugal deformation

WKB approximation

Principle

- wave-like solutions: $\Psi = \Re\left\{A(\vec{x})e^{i[\Phi(\vec{x})-\omega t]}\right\}$
- $\lambda \sim \|\vec{\nabla}\Phi\|^{-1} \ll$ length scale of the variations of the background L
- $\Phi = \Lambda(\Phi_0 + \frac{1}{\Lambda}\Phi_1 + \ldots)$ and $A = A_0 + \frac{1}{\Lambda}A_1 + \ldots$, with $\Lambda = L/\lambda$

Results ($\vec{k} = \vec{\nabla}\Phi_0$)

- $\omega = \mathcal{O}(\Lambda)$: acoustic waves $\omega = kc_s$
- $\omega = \mathcal{O}(1)$: gravito-inertial waves $k^2 = \frac{f^2}{\omega^2}{k_z}^2 + \frac{{N_0}^2}{\omega^2}{k_\perp}^2$
- in both cases, C is dominant near the surface: back-refraction of waves

Eikonal equation

$$\omega^2 = \frac{f^2{k_z}^2 + {N_0}^2{k_\perp}^2 + f^2\cos^2\Theta{k_c}^2}{k^2 + {k_c}^2}$$

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Ray dynamics equations

Motivation

- the eikonal equation is a PDE for Φ_0
  - we can solve it as a PDE (but it is expensive)
  - we can search for solutions along a certain path (=ray path)
- we need equations governing the ray path and the evolution of $\vec{k}$ along it

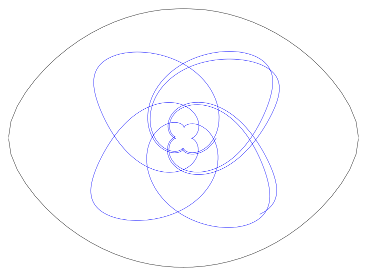
Hamiltonian formalism

- $\omega = \omega(\vec{x}, \vec{k})$
- ray path defined by the group velocity

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial \omega}{\partial k_i} \qquad (k_i = \partial \Phi / \partial x_i)$$

- conservation of ω requires

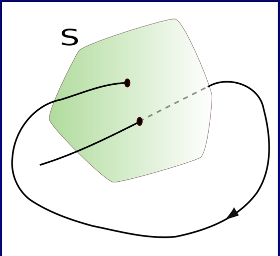
$$\frac{\mathrm{d}k_i}{\mathrm{d}t} = -\frac{\partial \omega}{\partial x_i}$$



Visualising the phase space with Poincaré surfaces of section (PSS)

General definition and properties

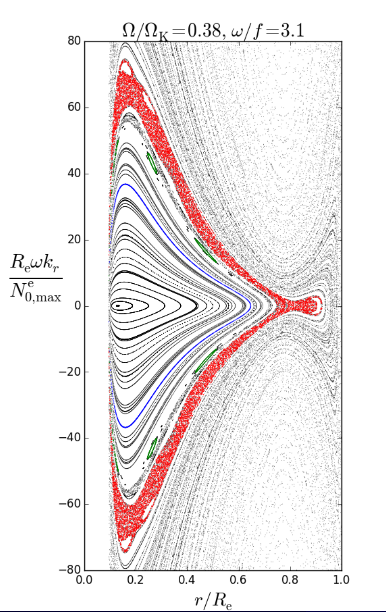
- N degrees of freedom ⇒ 2N-dimensional phase space
- PSS = intersection of all trajectories with a given (2N-1)-dimensional surface



Application

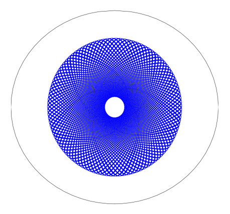
- $N = 2 \Rightarrow$ PSSs are 2D
- intersecting surface: equatorial plane $\theta = \pi/2 \rightarrow (r, k_r)$ coordinates
- PSSs at different frequencies allow us to scan the phase space

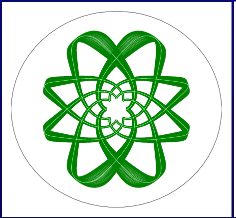
Types of structures

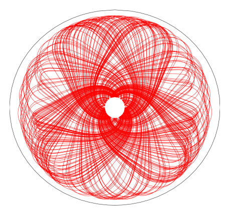


3 types of structures = 3 families of rays

- pseudo-integrable structures
- island chains around periodic orbits
- chaotic zones



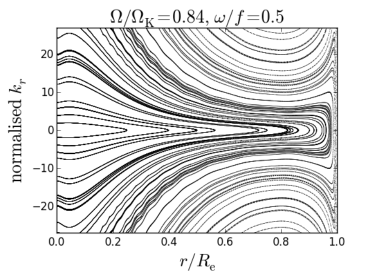




Low-frequency regime

Sub-inertial regime: $\omega < f$

- waves are trapped near the equatorial plane
- propagation possible near the centre
- mainly pseudo-integrable curves
- vicinity of an integrable system



Traditional approximation?

- neglect the horizontal component of the rotation vector and centrifugal deformation
- the system is integrable and separable...
- ...but waves cannot propagate near the centre

We need to go beyond the traditional approximation

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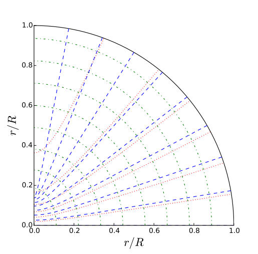
Low-frequency approximation

Coordinates associated to the principal frame

- avoid cross terms in the eikonal equation

$$\mathrm{d}\beta = \mathrm{d}s \sin\alpha + \mathrm{d}z \cos\alpha$$
$$\mathrm{d}\gamma = \mathrm{d}s \cos\alpha - \mathrm{d}z \sin\alpha$$

- $N_0^2 \sin[2(\alpha - \Theta)] = f^2 \sin 2\alpha$



Equatorial trapping

- $\cos^2\delta = \frac{N_0^2}{N_0^2+f^2}\cos^2\Theta \ll 1$
- near the equator, iso-contours of γ and δ are very close
- $k_\gamma \simeq \frac{k_\delta}{\zeta}$, where $\zeta = r\frac{\sqrt{N_0^2+f^2}}{N_0}$

Ray dynamics

Final form of the eikonal equation

$$\omega^2 = \frac{f^2 \cos^2\delta \left(k_\beta^2 + k_c^2\right) + \left(N_0^2 + f^2\right) \frac{k_\delta^2 + m^2}{\zeta^2}}{k^2 + k_c^2}$$

Ray dynamics equations

- further approximation: N_0, ζ, and k_c depend only on β

$$\frac{d\beta}{dt} = \frac{\left(f^2\cos^2\delta - \omega^2\right) k_\beta}{\omega(k^2 + k_c^2)}$$

$$\frac{dk_\beta}{dt} = \frac{\left\{\left[(N_0^2)'\zeta^2 - (\zeta^2)'\left(N_0^2 + f^2 - \omega^2\right)\right] \frac{k_\delta^2 + m^2}{\zeta^4} + (k_c^2)'\left(f^2\cos^2\delta - \omega^2\right)\right\}}{\omega(k^2 + k_c^2)}$$

$$\frac{d\delta}{dt} = \frac{\left(N_0^2 + f^2 - \omega^2\right) k_\delta}{\omega\zeta^2(k^2 + k_c^2)}$$

$$\frac{dk_\delta}{dt} = \frac{f^2 \sin\delta\cos\delta\left(k_\beta^2 + k_c^2\right)}{\omega(k^2 + k_c^2)}$$

Separable dynamics

New invariant: integrable system

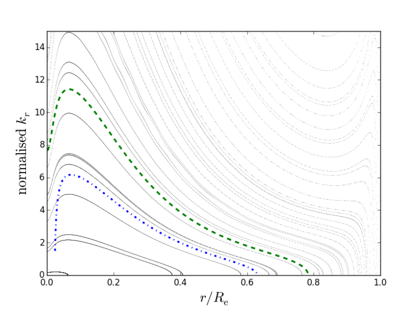
$$\chi = \frac{N_0^2 + f^2 \sin^2 \delta}{\zeta^2 (k^2 + k_c^2)} \qquad \left(\Rightarrow \lambda = \frac{k_\delta^2 + m^2}{1 - \frac{f^2}{\omega^2} \cos^2 \Theta} \right)$$

Separation of variables

$$k_\beta^2 + k_c^2 = \frac{N_0^2 + f^2 - \omega^2}{\zeta^2 \chi}$$

$$k_\delta^2 = \frac{\omega^2 - f^2 \cos^2 \delta}{\chi} - m^2$$

- reproduces quite well the structure of low-frequency PSSs
- the traditional approximation fails near the centre



Outline

Quantisation

EBK quantisation technique

$$\int_{\mathcal{C}} \vec{k} \cdot \mathrm{d}\vec{x} = 2\pi \left(p + \frac{\varepsilon}{4} \right)$$

- $\mathcal{C}$ is a closed contour
- p and ε are integers
- ε: Maslov index (accounts for phase-shifts due to boundary conditions)

Two independent contours → non-linear system to solve

$$\int_0^{r_\mathrm{s}} \sqrt{\frac{N_0^2 + f^2 - \omega^2}{\zeta^2 \chi} - k_\mathrm{c}^2} \mathrm{d}r = \frac{\pi}{2} \left(\tilde{n} + \frac{1}{2} \right) \qquad (\tilde{n} \sim 2n \text{ when } n \gg 1)$$

$$\int_{\delta_\mathrm{c}}^{\frac{\pi}{2}} \sqrt{\frac{\omega^2 - f^2 \cos^2 \delta}{\chi} - m^2} \mathrm{d}\delta = \frac{\pi}{2} \left(\tilde{\ell} + \frac{1}{2} \right) \qquad (\tilde{\ell} = \ell_\mu - 1)$$

with $\delta_\mathrm{c} = \arccos \frac{\sqrt{\omega^2 - m^2 \chi}}{f}$

Period spacings

Mode frequencies (in the co-rotating frame)

$$\omega^2 \simeq \frac{(2\tilde{\ell}+1) f \int_0^{r_s} \frac{N_0}{r} \mathrm{d}r}{\pi \left(n+\frac{1}{2}\right)} + m^2 \frac{\left(\int_0^{r_s} \frac{N_0}{r} \mathrm{d}r\right)^2}{\pi^2 \left(n+\frac{1}{2}\right)^2}$$

Axisymmetric modes ($m = 0$)

$$\Delta\Pi \simeq \frac{\pi^{3/2}}{\sqrt{2(n+1)(2\tilde{\ell}+1)\Omega \int_0^{r_s} \frac{N_0}{r} \mathrm{d}r}}$$

- scale as $1/\sqrt{n+1}$
- give access to $\Omega \int_0^{r_s} \frac{N_0}{r} \mathrm{d}r$

Non-axisymmetric modes ($m \neq 0$)

$$\Delta\Pi \simeq \frac{1}{m^2} \sqrt{\frac{2\pi(2\tilde{\ell}+1) \int_0^{r_s} \frac{N_0}{r} \mathrm{d}r}{\Omega^3 (n+1)^3}}$$

- scale as $(n+1)^{-3/2}$
- give access to $\int_0^{r_s} \frac{N_0}{r} \mathrm{d}r / \Omega^3$

Summary and prospects

Two different kind of spacings

- if both are measured, Ω and $\int_0^{r_s} \frac{N_0}{r} \mathrm{d}r$ can be extracted directly
- if using stellar models, $\int_0^{r_s} \frac{N_0}{r} \mathrm{d}r$ can be computed: only one kind needed for Ω

Beyond the traditional approximation

- new prescription formally similar to the traditional approximation
- more accurate near the centre
- not limited to spherical models

Further developments

- comparison with numerically computed modes
- use it to interpret observed oscillation spectra
- generalisation to differential rotation?

Thank you.

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