Parametric resonances in periodically perturbed dynamo models

André Giesecke & Frank Stefani
Helmholtz-Zentrum Dresden-Rossendorf

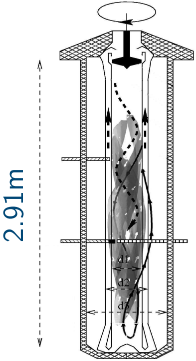
Workshop
Freiburg, May 7th - 9th 2018

Dynamo action and cosmic magnetic fields

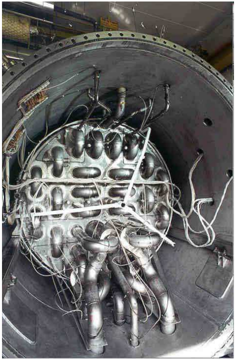
- cosmic magnetic fields can be observed on all scales: asteroids, moons, planets, stars, galaxies, ...
- magnetic field generation process:
  homogenous dynamo: transfer of kinetic energy of a flow of a conductive fluid into magnetic energy
  $\Rightarrow$ numerical, theoretical and experimental studies on fluid flow driven magnetic field generation
- this talk: focus on experimental dynamo action, but motivation also by natural systems
- response of system with impact by external periodic perturbation
- dynamo efficiency (growth rate, threshold), phase

Fluid Flow Driven Dynamo Experiments

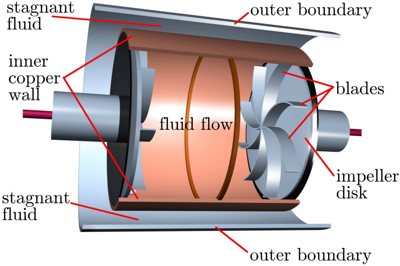
- complex problem because induced currents do not flow along fixed path (like in technical dynamo) but are determined by the nature of the induced electromotive force $\mathcal{E} \propto \mathbf{u} \times \mathbf{B}$
- three successful experiments with fluid flow driven dynamo action



Riga



Karlsruhe

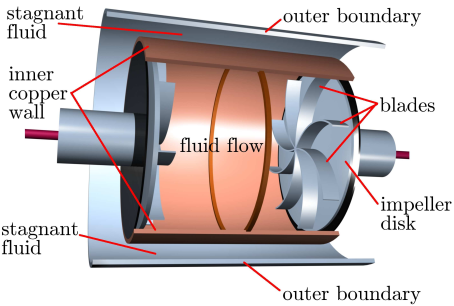


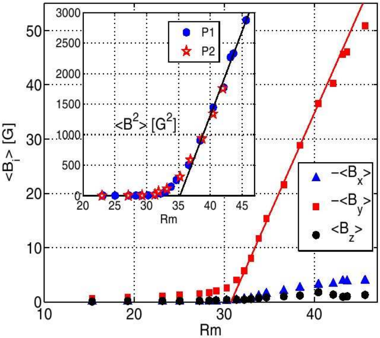
VKS/Cadarache

- precession dynamo experiment is under construction at HZDR

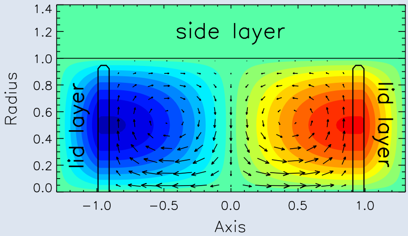
Example: Von-Kármán-Sodium (VKS) dynamo

from Monchaux et al., PRL, 2007



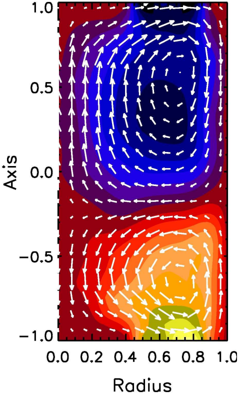


- flow of liquid sodium driven by two counterrotating impellers
- mean flow structure: two poloidal cells, two toroidal cells ($S2T2$)
- dynamo at $\mathrm{Rm}^c = \mathcal{U}_{\max} R/\eta \approx 32$ but soft iron impellers $\mu_r \approx 65$

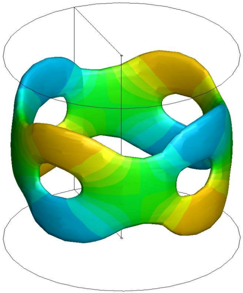


- water experiment at Universidad de Navarra (Pamplona) used for estimation of flow properties

turbulent flow in water experiment



mean axi-symmetric flow



non-axisymmetric $m = 2$ perturbation

Dynamo action and periodic perturbations

- impact of non-axisymmetric perturbations on dynamo action driven by a large scale axisymmetric flow

  $\Rightarrow$ beneficial or obstructive

- possible realisation in natural or experimental dynamos
  - convection driven dynamos with impact from tidal forcing ($m = 2$) or precession ($m = 1$)
  - azimuthally drifting vortices in cylindrical flow with von-Kármán like driving (Giesecke et al. PRE 2012)
  - 'swing-excitation' caused by density waves in spiraling galaxies (Chiba & Tosa, MNRAS 2002)
- synchronization of dynamo cycles with external impact of periodic distortion (Stefani et al, Solar Physics 2016)

Kinematic models

induction equation

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B})$$

$$\mathbf{B} \sim \mathbf{B}_0(\mathbf{r}) e^{\lambda t} \Rightarrow \lambda \mathbf{B} = \mathcal{M}\mathbf{B}$$

- prescribed velocity field $\mathbf{u}(\mathbf{r}, \mathbf{t})$
- linear problem $\Rightarrow$ eigenvalues λ
- no magnetic backreaction

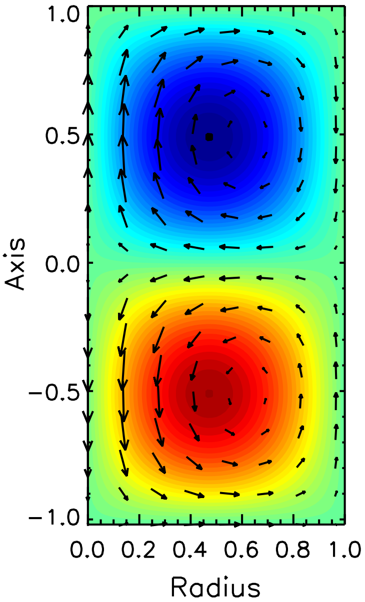
Two different approaches for solution

(1) timestepping with hybrid Finite Volume/Boundary Element Method
$\Rightarrow$ calculation of eigenvalues $\Rightarrow \lambda = \gamma + i\omega$
with growth rates γ and frequencies ω

(2) simple analytic model for field amplitude
$\Rightarrow$ periodic velocity perturbation $\mathbf{u} = \mathbf{u}_0^{\mathrm{axi}} + \epsilon \mathbf{u}_1^{\mathrm{nonaxi}}(t)$
$\Rightarrow$ Floquet-theorem: $\mathbf{B} \sim P(t) e^{Rt}$ with $P(t)$ same periode as perturbation and constant matrix R

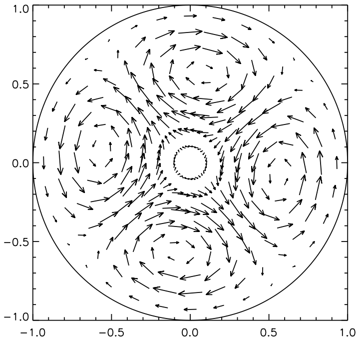
Simplified velocity model

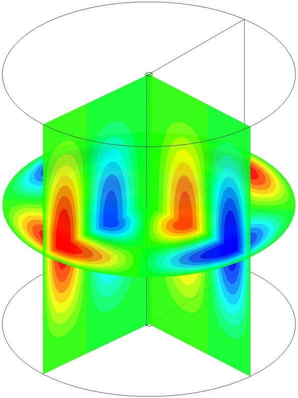
- Beltrami like flow $\nabla \times \mathbf{u} \propto \mathbf{u}$
  $\Rightarrow$ helicity maximizing
  $\Rightarrow$ well suited for dynamo action
- meridional flow: $\mathbf{u} = \nabla \times \Psi$ with
  $\Psi = J_1(\alpha r) \sin\left(\frac{2\pi z}{H}\right) \hat{\mathbf{e}}_\varphi$
- toroidial flow is given by
  $u_\varphi = -\sqrt{\alpha^2 + \left(\frac{2\pi}{H}\right)^2} J_1(\alpha r) \sin\left(\frac{2\pi z}{H}\right)$
- J_1 cylindrical Bessel function
  $\alpha = 3.8317$ (first zero of J_1)
  $\Rightarrow \nabla \times \mathbf{u} = -\sqrt{\alpha^2 + \left(\frac{n\pi}{H}\right)^2}\mathbf{u}$
- flow consists of 2 toroidal flow cells with different orientation and 2 recirculating cells in the meridional plane (somehow related to the mean flow in the VKS dynamo)



Modelling the velocity perturbation

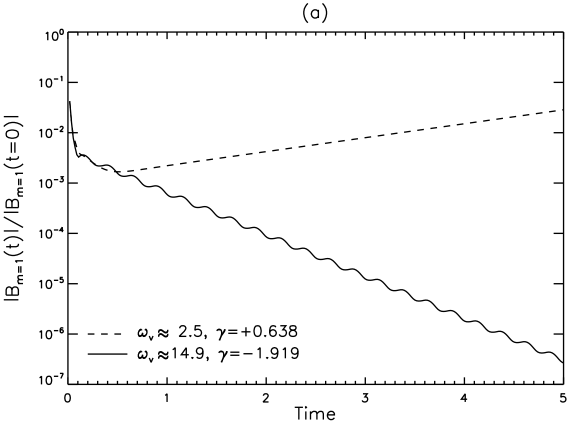
$$\mathbf{u}'(\mathbf{r},t) = \nabla \times \mathbf{A}\cos(m(\varphi+\omega)t) \text{ with } \mathbf{A} = V_a r\,[\cos(2\pi r)-1]\cos(2\pi z)\hat{\mathbf{e}}_z$$





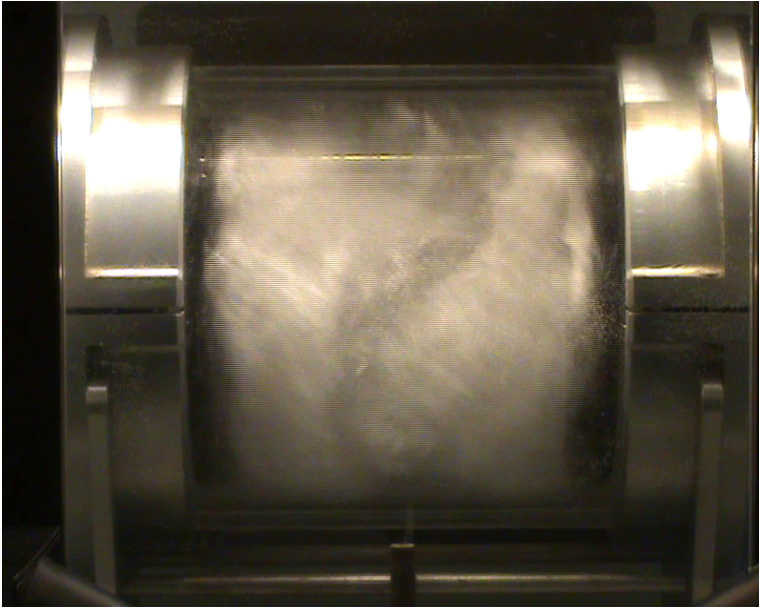
- vortex-like structure along the axis with azimuthal wavenumber $m = 2$
- azimuthal propagation of perturbation with drift frequency ω

Numerical Results: temporal evolution of **B**

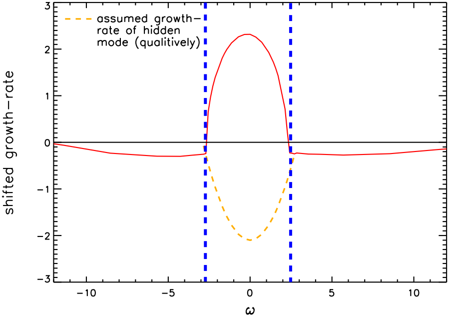


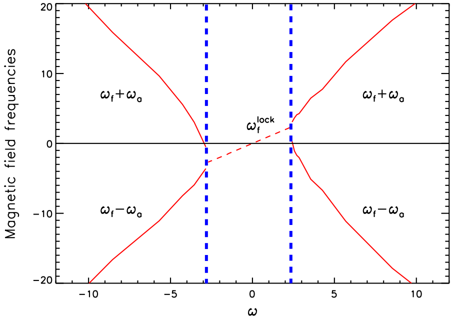
- temporal behavior of magnetic eigenmode depends on azimuthal drift of the non-axisymmetric perturbation
- "fast" drift ⇒ decaying solution with amplitude modulation
- "slow drift" ⇒ growing solution and no amplitude modulation

Visualization of the behavior



Growth rates and frequencies



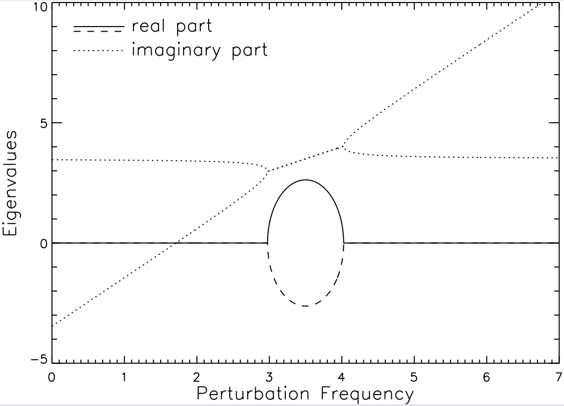


Coalescence of Eigenmodes

- merging of eigenmodes ⇒ parametric resonance for $|\omega| \lesssim 2\omega_0$
- exceptional points (degeneration of eigenvalues and eigenfunctions)
- frequency locking within resonant regime
- similar to mechanical systems subject to periodic distortions

Analogy with periodically perturbed oscillator

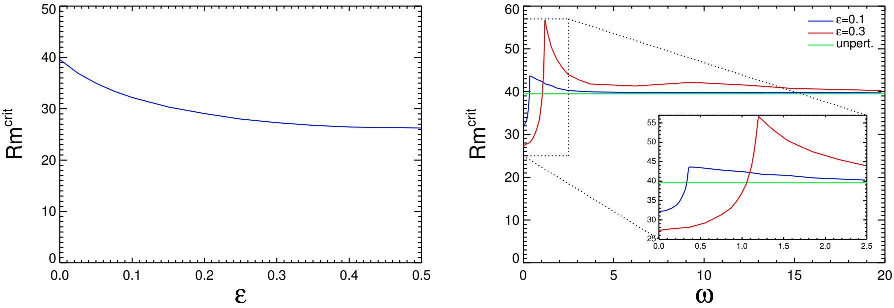
Matthieu Equation: $\ddot{Q}+\omega^2(t)Q=0$ with $\omega(t)=\omega_0(1+\epsilon\cos(\widetilde{\omega}t))$



$\Rightarrow$ growthrates in resonant regime: $\gamma=\pm\sqrt{(\epsilon\omega_0)^2-(\widetilde{\omega}-2\omega_0)^2}$
outside resonance: $\omega=0.5\widetilde{\omega}\pm\sqrt{(\widetilde{\omega}-2\omega_0)^2-(\epsilon\omega_0)^2}$

$\Rightarrow$ within resonant regime solution **locks** to perturbation frequency $\omega=\widetilde{\omega}$

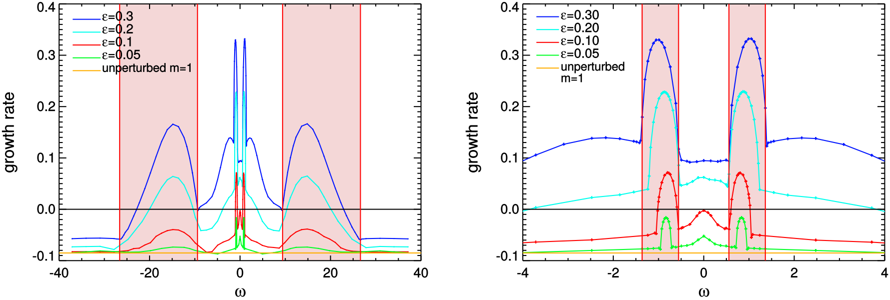
Reduction of threshold



- "asymptotic" behavior for increasing perturabtion amplitude
- reduction from $\mathrm{Rm}^{c} = 39$ to $\mathrm{Rm}^{c} = 26$
- optimum depends on azimuthal drift of unperturbed eigenmode (here strongest reduction for "standing" vortices with $\omega = 0$)

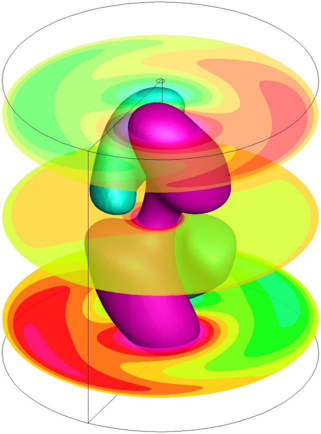
More complex solutions possible...

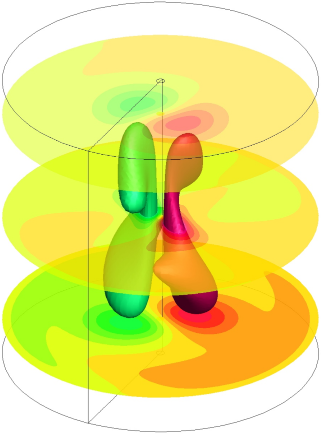
non-drifting basic state but amplitude modulation of magnetic field



- sharp peaks for $\Omega_{\mathrm{p}} \approx 2\Omega_0 \Rightarrow$ parametric resonance
- location of maximum depends on amplitude of perturbation
- broader peaks for larger frequencies (but no resonance)

Temporal behavior of magnetic eigenmode





Low dimensional model

- assume axisymmetric flow $\mathbf{U}_0$ and a non-axisymmetric periodic perturbation with azimuthal wave number $\widetilde{m}$ and frequency ω

$$\mathbf{U}(\mathbf{r},t) = \mathbf{U}_0(r,z) + \epsilon\left[\mathbf{u}_{\widetilde{m}}(r,z)e^{i(\widetilde{m}\varphi+\omega t)} + \mathbf{u}_{-\widetilde{m}}(r,z)e^{-i(\widetilde{m}\varphi+\omega t)}\right]$$

- reduction of induction equation into a system of equations for the amplitude of azimuthal field modes

$$\mathbf{B} = \sum_{-M}^{M} \hat{b}_m(t)\mathbf{b}_m(r,z)e^{im\varphi}$$

Assumptions

- modes are modulated by simple temporal varying amplitude
- consider only leading eigenmode for each azimuthal wave number (i.e. only one single mode is close to be unstable)

but linear operator in induction equation is non-normal

$\Rightarrow$ in general these assumptions are not correct

Floquet approach

$\mathbf{B} = \sum_{-M}^{M} \hat{b}_m(t)\mathbf{b}_m(r,z)e^{im\varphi} \Rightarrow$ induction equation written as matrix equation

$$\frac{d}{dt}B(t) = A(t)B(t)$$

$$A = \begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\
\cdots & 0 & \alpha^*_{3,5}f^*_t & \alpha^*_{3,3} & \alpha^*_{3,1}f_t & 0 & \cdots & & & \\
 & \cdots & 0 & \alpha^*_{1,3}f^*_t & \alpha^*_{1,1} & \alpha^*_{1,-1}f_t & 0 & \cdots & & \\
 & & \cdots & 0 & \alpha_{1,-1}f^*_t & \alpha_{1,1} & \alpha_{1,3}f_t & 0 & \cdots & \\
 & & & \cdots & 0 & \alpha_{3,1}f^*_t & \alpha_{3,3} & \alpha_{3,5}f_t & 0 & \cdots \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}$$

- diagonal elements: $\alpha_{j,j}$ are the growth rates of the unperturbed case
- off-diagonal parameters $\alpha_{m,m\pm2} \Rightarrow$ interaction of adjacent modes
- time dependence: $f_t = \epsilon e^{i\omega t}$

Floquet approach

(1) solution of $\frac{d}{dt}B(t) = A(t)B(t)$ with a T-periodic matrix $A(t){=}A(t{+}T)$ and a $n-$dimensional vector B is given by $B(t) = P(t)e^{Rt}$ with a T-periodic invertible matrix $P(t) = P(t+T)$ and a constant matrix R

(2) it is always possible to find a transformation $B(t) = P(t)X(t)$ such that $\frac{d}{dt}X(t) = RX(t)$ and $R = e^{-iDt}Ae^{iDt} - iD$ is constant

(3) in our particular case we can write $A(t) = e^{iD_\omega t}\hat{A}e^{-iD_\omega t}$ with $\hat{A}$ the matrix A without time modulation $e^{\pm i\omega t}$ and

$$D_\omega = \begin{pmatrix} -\frac{M}{2}\omega & 0 & & & \\ 0 & -\frac{M-2}{2}\omega & 0 & & \\ & 0 & \ddots & 0 & \\ & & 0 & \frac{M-2}{2}\omega & 0 \\ & & & 0 & \frac{M}{2}\omega \end{pmatrix}$$

$\Rightarrow$ $R = \hat{A} - iD_\omega$ so that $\frac{dX}{dt} = \left(\hat{A} - iD_\omega\right)X$ with solutions $X = X_0 e^{\widetilde{\sigma}t}$

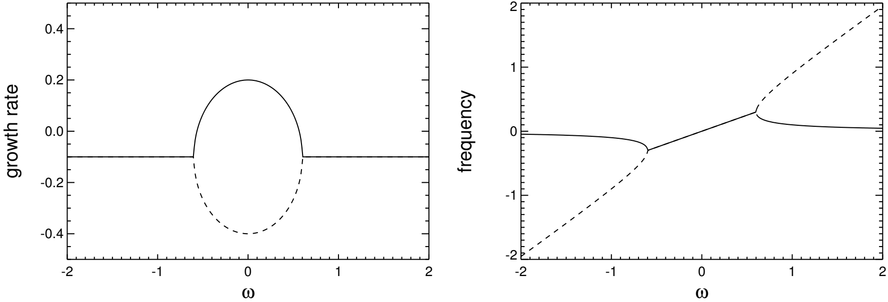
$\Rightarrow$ eigenvalues $\widetilde{\sigma}$ are roots of characteristic equation $\left|\hat{A} - iD_\omega - \widetilde{\sigma}\mathbb{I}\right| = 0$

Example: Truncation at $M = 1$

$$\begin{pmatrix} \frac{d}{dt}\hat{b}_{-1} \\ \frac{d}{dt}\hat{b}_{1} \end{pmatrix} = \begin{pmatrix} \alpha^* & \epsilon e^{-i\omega t}\gamma^* \\ \epsilon e^{i\omega t}\gamma & \alpha \end{pmatrix} \begin{pmatrix} \hat{b}_{-1} \\ \hat{b}_{1} \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} \hat{A} &= \begin{pmatrix} \alpha^* & \epsilon\gamma^* \\ \epsilon\gamma & \alpha \end{pmatrix} \\ D_\omega &= \begin{pmatrix} -\omega/2 & 0 \\ 0 & \omega/2 \end{pmatrix} \end{aligned}$$

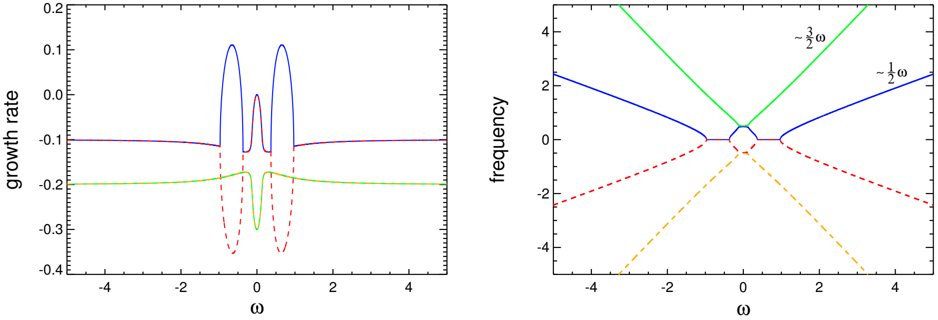
characteristic equation

$$\left(\alpha^* + i\frac{\omega}{2} - \widetilde{\sigma}\right)\left(\alpha - i\frac{\omega}{2} - \widetilde{\sigma}\right) - \epsilon^2\left|\gamma\right|^2 = 0 \Rightarrow \widetilde{\sigma}_\pm = \alpha_r \pm \frac{1}{2}\sqrt{4\epsilon^2|\gamma|^2 - (2\alpha_i - \omega)^2}$$



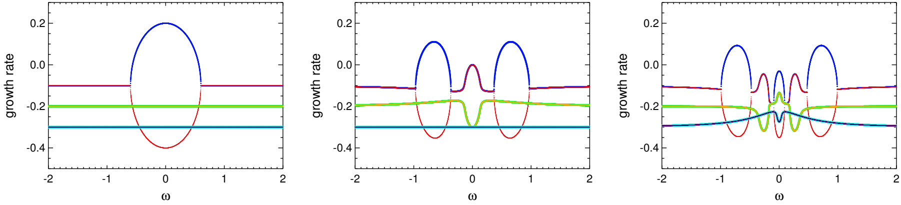
Perturbed system

higher truncation order: $M = 3$

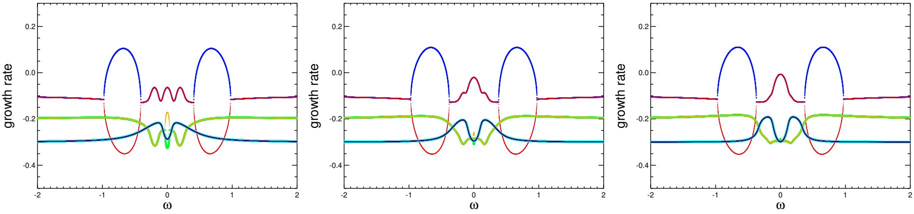


- models become more complex for higher truncation order
- interaction $m = 1$ and $m = -1 \Rightarrow$ parametric resonance
  interaction $m = 1$ and $m = 3 \Rightarrow$ amplification, no frequency locking

Perturbed system



- incvreasing order of truncation ($M = 1, 3, 5$) leads to increasing complexity $\Rightarrow$ coupling must decrease for convergence of model



Summary, Conclusions & Outlook

- external perturbation may have a significant impact on dynamo action driven by a large scale axisymmetric flow
- mechanism: coupling of different eigemodes
  $\Rightarrow$ enhancement of dynamo efficiency
- complex dependence on frequency and amplitude
- Natural example: precession driven flows in case of a triadic resonance of inertial modes but realization not very probable
- frequency locking occurs in case of parametric resonance but amplificatin/enhancement of growth rates possible without synchronization
- frequency locking requires $\Omega_{\mathrm{p}} = 2\Omega_0$, i.e. adoption of external perturbation frequency to internal frequency of system
  $\Rightarrow$ different from non-linear synchronization in dynamo model presented by Frank